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Sequential detection of transient changes in stochastic-dynamical systems *

Titre: Détection séquentielle de changements transitoires dans des systèmes stochastiques - dynamiques

Van Long Do[1] , Lionel Fillatre[2] and Igor Nikiforov[1]

Abstract: This paper deals with the problem of detecting transient changes in stochastic-dynamical systems. A statistical observation model which depends on unknown system states (often regarded as the nuisance parameter) is developed. The negative impact of nuisance parameter is then eliminated from the observation model by utilizing the invariant statistics. The Variable Threshold Window Limited CUmulative SUM (VTWL CUSUM) test, previously developed for independent observations, is adapted to the novel observation model. Taking into account the transient change detection criterion, minimizing the worst-case probability of missed detection subject to an acceptable level of the worst-case probability of false alarm within a given time period, the thresholds of the VTWL CUSUM test are optimized. It is shown that the optimized VTWL CUSUM algorithm is equivalent to the Finite Moving Average (FMA) detection rule. A numerical method for estimating the probability of false alarm and missed detection is proposed. The theoretical results are applied to the problem of cyber/physical attack (stealing water from a reservoir) detection on a simple Supervisory Control and Data Acquisition (SCADA) water distribution system.

Résumé : Cet article s'intéresse au problème de détection de changements transitoires dans des systèmes stochastiques et dynamiques. Le modèle d'observation statistique étudié dépend de l'état inconnu du système considéré comme un paramètre de nuisance. Ce paramètre de nuisance est éliminé en utilisant la technique, bien connue dans la communauté du diagnostic automatique, de la projection des observations dans l'espace de parité. L'algorithme de la Somme Cumulée à Fenêtre Limitée et Seuils Variables (VTWL CUSUM) est adapté au modèle d'observation utilisé. Le critère de détection de changement transitoire étudié vise à minimiser la pire probabilité de détection manquée sous la contrainte que la pire probabilité de la fausse alarme soit bornée pendant une période de longueur donnée. Les seuils de l'algorithme sont optimisés pour obtenir la meilleure performance. Il est montré que l'algorithme VTWL CUSUM optimal est équivalent à l'algorithme de la Moyenne Glissante Finie (FMA). Une méthode numérique est proposée pour estimer les probabilités de fausse alarme et de détection manquée. Enfin, les résultats théoriques sont appliqués à la détection d'attaques cyber-physiques, dans un système de distribution d'eau potable, qui ont pour but de voler l'eau d'un réservoir.

Keywords: transient change detection, stochastic-dynamical systems, criterion of optimality, CUSUM-based algorithm, probability of false alarm, probability of missed detection, cyber/physical attacks
Mots-clés : détection de changements transitoires, systèmes stochastiques et dynamiques, critère d'optimalité, algorithme CUSUM, probabilité de fausse alarme, probabilité de détection manquée, attaques cyber-physiques.
AMS 2000 subject classifications: 62C20, 62L15, 62P30

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[1] Laboratoire LM2S, Université de Technologie de Troyes, France
E-mail: van_long.do@utt.fr and E-mail: igor.nikiforov@utt.fr
[2] Laboratoire I3S, Université de Nice Sophia Antipolis, France
E-mail: lionel.fillatre@i3s.unice.fr

1. Introduction

The problem of detecting abrupt changes in stochastic systems has many important applications, including fault detection in complex technical systems, on-line monitoring of safety-critical infrastructures, detection of signals with unknown arrival time in radar and sonar signal processing, and segmentation of signals.

The sequential change detection problem, including the quickest change detection and the transient change detection, consists in calculating the stopping time T at which the change is detected, based on the sequence of observations. The traditional quickest change detection problem deals with an abrupt change in dynamic-stochastic systems, where the post-change period is assumed to be infinitely long. The average detection delay should be as small as possible subject to an acceptable level of false alarms. There is an extensive literature in the theory of quickest change detection: see, for example, Basseville and Nikiforov (1993); Lai (1995, 1998, 2001); Poor and Hadjiliadis (2009); Tartakovsky and Moustakides (2010); Polunchenko and Tartakovsky (2012); Tartakovsky et al. (2014). In the non-Bayesian framework, where the change time is unknown but non-random, some optimal algorithms with respect to (w.r.t.) different criteria of optimality have been introduced in Lorden (1971); Moustakides (1986); Lai (1998); Tartakovsky (2005). On the other hand, some optimality criteria as well as optimal detection rules under the Bayesian framework, where the change time is unknown and random, can be found in Shiryaev (1963); Roberts (1966); Pollak (1985); Tartakovsky and Moustakides (2010).

In contrast to the quickest change detection problem, in the transient change detection, the post-change period is usually short. The traditional quickest change detection criterion minimizing the average detection delay subject to an acceptable level of false alarms is not suitable for a short post-change period. Here, we wish to minimize the probability of missed detection subject to an acceptable level of false alarms. The optimality criterion involving the minimization of the worst-case (conditional) probability of missed detection subject to an upper bound on the worst-case probability of false alarm within a given time period has been proposed in Bakhache and Nikiforov (2000) and Guépié et al. (2012b). A sub-optimal solution w.r.t. this criterion in the case of Gaussian independent observations has been introduced in Guépié et al. (2012b).

In safety-critical infrastructures such as the electric power systems, water distribution systems or gas pipelines, it is desirable to detect the abnormal situations with the detection delay upper bounded by a prescribed value. The above mentioned transient change detection criterion is well-adapted to such situations. Examples of such safety-critical applications include, among others, the radar and sonar detection Streit and Willett (1999), the navigation system integrity monitoring Bakhache and Nikiforov (2000), the water distribution system monitoring Guépié et al. (2012b), or cyber attacks on networked control systems Huang et al. (2009).

The majority of such complex technical systems can be modeled by stochastic-dynamical systems. Often it can be a discrete-time state space model, where the transient changes are modeled as additive signals of short duration. By pursuing the work of Guépié et al. (2012b,a), the goal of this paper is to propose a sub-optimal algorithm for detecting the transient changes in stochastic-dynamical systems. The contribution of this paper is threefold :

– The generalization of the previously developed Variable Threshold Window Limited CUmulative SUM (VTWL CUSUM) test to a discrete-time state space model with nuisance parameters and additive transient changes.

– The optimization of the proposed VTWL CUSUM test w.r.t. the above mentioned criterion. It is shown that the optimized VTWL CUSUM test leads to the FMA detection rule. A numerical method for evaluating the statistical performance of the proposed algorithm is also provided.
– The application of the optimized VTWL CUSUM test to the detection of cyber/physical attacks on a simple SCADA water distribution network.

This paper is organized as follows. The problem statement is given in Section 2. The design of the VTWL CUSUM algorithm for detecting transient changes of known profiles in stochastic-dynamical systems is described in Section 3. In Section 4, the statistical performance and the optimization of the proposed algorithm are investigated. The theoretical findings are applied to the detection of cyber/physical attacks on a simple SCADA water distribution network and they are compared with the results of Monte Carlo simulation in Section 5. Some concluding remarks and perspectives are given in Section 6.

2. Problem statement

Firstly, we shortly recall the problem of sequential detection of abrupt changes in random signals. Then, we introduce the problem of transient change detection. Finally, the state space model with nuisance parameters and additive transient changes is considered.

2.1. Quickest change detection

Let $\{y_k\}_{k\geq 1}$ be an independent random sequence observed sequentially. Under normal operation, the observations $y_1,\cdots,y_{k_0-1}$ follow a distribution $\mathscr{P}_0$ (resp. the cumulative distribution function (c.d.f.) F_0 and the probability density function (p.d.f.) p_0). From an unknown change time k_0, the observations $y_{k_0},y_{k_0+1},\cdots,y_\infty$ follow another distribution $\mathscr{P}_1 \neq \mathscr{P}_0$ (resp. the c.d.f. F_1 and the p.d.f. p_1), corresponding to the abnormal behavior. The statistical model for the quickest change detection problem is described as follows:

$$y_k \sim \begin{cases} \mathscr{P}_0 & \text{if } k < k_0 \\ \mathscr{P}_1 & \text{if } k \geq k_0 \end{cases}, \tag{1}$$

where k_0 is the unknown change time.

A quickest change detection algorithm should calculate the stopping time T at which the change is decided. The mean detection delay should be as small as possible subject to an acceptable level of false alarms. Let $\mathscr{P}^{k_0}$ be the joint distribution of the independent observations $y_1,y_2,\dots,y_{k_0-1},y_{k_0},\dots,y_\infty$ when $y_i \sim \mathscr{P}_0$ for $1 \leq i \leq k_0-1$ and $y_i \sim \mathscr{P}_1$ for $i \geq k_0$. Let $\mathbb{E}_{k_0}$ (resp. $\mathbb{E}_0$) and $\mathbb{P}_{k_0}$ (resp. $\mathbb{P}_0$) be the mathematical expectation and probability w.r.t. the distribution $\mathscr{P}^{k_0}$ (resp. $\mathscr{P}_0 = \mathscr{P}^\infty$).

The first optimality results in the non-Bayesian approach were obtained in Lorden (1971), where the author proposed to minimize the worst-case (conditional) mean detection delay

$$\overline{\mathbb{E}}(T) = \sup_{k_0\geq 1} \operatorname{esssup} \mathbb{E}_{k_0}\left[(T-k_0+1)^+ | y_1,y_2,\cdots,y_{k_0-1}\right] \tag{2}$$

among all stopping times T satisfying $\mathbb{E}_0(T) \geq \gamma$, where $(x)^+ = \max(0,x)$, $\mathbb{E}_0(T)$ is the average run length to false alarm (ARL2FA) and $\gamma > 0$ is the prescribed value for ARL2FA. The following CUSUM detection rule, which was first introduced by Page (1954):

$$T_{CS} = \inf\left\{k \geq 1 : \max_{1\leq i\leq k} S_i^k \geq h\right\}; \quad S_i^k \triangleq \sum_{t=i}^{k} \log \frac{p_1(y_t)}{p_0(y_t)} \tag{3}$$

was proved to be asymptotically optimal (as $\gamma \to \infty$) w.r.t. the Lorden's criterion, where h is the chosen threshold. The non-asymptotic optimality of the CUSUM test has been established in Moustakides (1986) and Ritov (1990).

However, as shown in Lai (1998), the requirement of having large values of the ARL2FA $\mathbb{E}_0(T)$ does not guarantee small values of the probability of false alarm $\mathbb{P}_0(l \leq T \leq l+m_\alpha-1)$ within a fixed size of time window m_α, for all $l \geq 1$. As a result, Lai (1998) proposed to replace traditional constraint on the ARL2FA $\mathbb{E}_0(T) \geq \gamma$ by the following constraint on the worst-case probability of false alarm within any time window of length m_α:

$$\sup_{l\geq 1} \mathbb{P}_0(l \leq T \leq l+m_\alpha-1) \leq \alpha, \tag{4}$$

where $\liminf m_\alpha/|\log\alpha| > \rho_{10}^{-1}$ but $\log m_\alpha = o(|\log\alpha|)$ as $\alpha \to 0$ and $\rho_{10} = \mathbb{E}_1[\log p_1(y)/p_0(y)]$ stands for the Kullback-Leibler information number between p_0 and p_1. Moreover, Lai (1998) has shown that the Window Limited CUSUM test, which was first introduced by Willsky and Jones (1976), minimizes the average detection delay $\mathbb{E}_{k_0}(T-k_0)^+$ over all stopping times T satisfying the criterion (4), uniformly in $k_0 \geq 1$.

2.2. *Transient change detection*

Unlike the quickest change detection problem, the post-change period in the transient change detection problem is assumed to be short. The following statistical model is used for describing transient changes in a stochastic system:

$$y_k \sim \begin{cases} \mathscr{P}_0 & \text{if } k < k_0 \\ \mathscr{P}_1 & \text{if } k_0 \leq k < k_0+L \\ \mathscr{P}_0 & \text{if } k \geq k_0+L \end{cases}, \tag{5}$$

where L denotes the change duration. As discussed in Guépié et al. (2012b), there are two types of transient change detection problem. The first type is the detection of suddenly arrived short signal of random unknown duration L, say an acoustic signature. The second type involves the safety-critical applications such as the integrity monitoring in navigation systems or the detection of cyber/physical attacks on SCADA systems. In such circumstances, the maximum permitted detection delay is *a priori* fixed to a prescribed value L. This value is calculated taking into account the gravity of a fault/attack undetectable during L unit of time and the detection of a change with the delay greater than L is then considered as missed. Therefore, the optimality criteria for the transient detection problem should favor a small probability of missed detection given an acceptable false alarm rate.

Let us assume that starting from now $\mathscr{P}^{k_0}$ be the joint distribution of the observations $y_1, y_2, \ldots, y_{k_0-1}, y_{k_0}, \ldots, y_\infty$ when y_k follows the transient change model described by (5). An analysis of different methods for detection of transient changes can be found in Guépié et al. (2012b). It is shown that the existing methods of transient change detection are mainly applicable to finite observation intervals, i.e. to *a posteriori* transient change detection. The only exclusions are Repin (1991); Han et al. (1999); Chen and Willett (2000); Bakhache and Nikiforov (2000); Premkumar et al. (2010). The traditional quickest change detection criterion involving minimization of the mean detection delay under constraint on the false alarm probability is used in Chen and Willett (2000) and Premkumar et al. (2010). Another criterion of optimality involving the minimization of the worst-case probability of missed detection $\sup_{k_0 \geq 1} \mathbb{P}_{k_0}(T - k_0 + 1 > L)$ subject to the constraint (4) has been proposed in Bakhache and Nikiforov (2000) under a non-Bayesian setting. The minimization of the probability of missed detection under constraint on the false alarm probability is considered in Premkumar et al. (2010) under a Bayesian setting.

Motivated by safety-critical applications, we use through this paper the criterion of optimality introduced in Guépié et al. (2012b), which involves the minimization of the worst-case conditional probability of missed detection (under the assumption that the change does not occur during the "preheating" period (i.e. $k_0 \geq L$))

$$\inf_{T \in C_\alpha} \left\{ \overline{\mathbb{P}}_{md}(T; L) = \sup_{k_0 \geq L} \mathbb{P}_{k_0}(T - k_0 + 1 > L | T \geq k_0) \right\} \tag{6}$$

among all stopping times $T \in C_\alpha$ satisfying

$$C_\alpha = \left\{ T : \overline{\mathbb{P}}_{fa}(T; m) = \sup_{l \geq L} \mathbb{P}_0(l \leq T < l + m - 1) \leq \alpha \right\} \tag{7}$$

where $\overline{\mathbb{P}}_{md}$ denotes the worst-case probability of missed detection and $\overline{\mathbb{P}}_{fa}$ stands for the worst-case probability of false alarm within any time window of length m.

Remark 1. *Let us discuss the motivation of criterion (6) – (7). If the traditional quickest change detection criterion, i.e., the (worst-case) mean detection delay, is used in the framework of safety-critical applications, then this criterion evaluates the (worst-case) weighted sum of timely detection delays (when $T - k_0 + 1 \leq L$) and latent detection delays (when $T - k_0 + 1 > L$). But, as it follows from the previous paragraphs, we are interested in minimizing the proportion (probability) of latent detections (when $T - k_0 + 1 > L$). Moreover, in the case of timely detections (resp. latent detections) the true delay has no significance. For this reason the traditional (worst-case) mean detection delay criterion is unacceptable for safety-critical applications and the minimization of the worst-case probability of latent detection (when $T - k_0 + 1 > L$) (6) should be used instead. Some difficulties in using the traditional quickest change detection criterion in the case of transient change detection are also discussed in Bakhache and Nikiforov (2000). Concerning the definition of the class C_α, see (7), the idea of Lai (1998) that the criterion involving the probability of false alarm within any time window is more stringent than the traditional ARL2FA criterion is used here, see Section 2.1.*

To detect the transient change (5), a Variable Threshold (VT) modification of the WL CUSUM test has been proposed in Guépié et al. (2012a); Guépié (2013). The stopping time of the VTWL

CUSUM test is :

$$T_{VTWL} = \inf\left\{k \geq L : \max_{k-L+1 \leq i \leq k}\left(S_i^k - h_{k-i+1}\right) \geq 0\right\}, \tag{8}$$

where T_{VTWL} is the stopping time, the variable thresholds $h_1, h_2, \cdots, h_L$ are used as tuning parameters to minimize the worst-case probability of missed detection $\sup_{k_0 \geq L} \mathbb{P}_{k_0}\left(T_{VTWL} - k_0 + 1 > L | T_{VTWL} \geq k_0\right)$ among all VTWL CUSUM tests $T_{VTWL} \in C_\alpha$ satisfying the constraint (7). It has been shown that the optimization of the VTWL CUSUM test leads to the following Finite Moving Average (FMA) test:

$$T_{FMA} = \inf\left\{k \geq L : \sum_{i=k-L+1}^{L} y_i \geq \widetilde{h}\right\}, \tag{9}$$

where $\widetilde{h}$ is the chosen threshold. This paper generalizes the previous works Guépié et al. (2012b,a); Guépié (2013) to the detection of transient changes in stochastic-dynamical systems.

2.3. *Transient change in stochastic-dynamical systems*

Let us consider a stochastic-dynamical system without process noise. Under normal operation, the system model can be described by a discrete-time state space form as follows:

$$\begin{cases} x_{k+1} &= Ax_k + Bu_k + Fd_k \\ y_k &= Cx_k + Du_k + Gd_k + \xi_k \end{cases}; \quad x_0 = \bar{x}_0, \tag{10}$$

where $x_k \in \mathbb{R}^n$ is the vector of system states with unknown initial conditions $x_0 = \bar{x}_0$, $u_k \in \mathbb{R}^m$ is the vector of control signals, $d_k \in \mathbb{R}^q$ is the vector of disturbances, $y_k \in \mathbb{R}^p$ is the vector of sensor measurements, $\xi_k \in \mathbb{R}^p$ is the vector of sensor noises, which is assumed to be a multivariate zero-mean Gaussian distribution $\xi_k \sim \mathcal{N}(0, R)$, where $R \in \mathbb{R}^{p \times p}$ is a known constant positive-definite matrix; the matrices $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$, $F \in \mathbb{R}^{n \times q}$, $C \in \mathbb{R}^{p \times n}$, $D \in \mathbb{R}^{p \times m}$, and $G \in \mathbb{R}^{p \times q}$ are assumed to be known.

Let us assume that a transient change occurs in the system, impacting, both, the state space equation and the sensor measurement equation during a short period $[k_0, k_0 + L - 1]$, where k_0 is an unknown change point and L is the transient change period, assumed to be known. The system model including the transient change is given by

$$\begin{cases} x_{k+1} &= Ax_k + Bu_k + Fd_k + B_a a_k \\ y_k &= Cx_k + Du_k + Gd_k + D_a a_k + \xi_k \end{cases}; \quad x_0 = \bar{x}_0, \tag{11}$$

where the matrices $B_a \in \mathbb{R}^{n \times s}$ and $D_a \in \mathbb{R}^{p \times s}$ are decided by the system architecture and the transient change vector $a_k \in \mathbb{R}^s$ can be modeled as follows:

$$a_k = \begin{cases} 0 & \text{if } k < k_0 \\ \theta_{k-k_0+1} & \text{if } k_0 \leq k < k_0 + L \\ 0 & \text{if } k \geq k_0 + L \end{cases}, \tag{12}$$

where $\theta_{k-k_0+1} \in \mathbb{R}^s$ for $k_0 \leq k \leq k_0+L-1$ is the change profile, which is assumed perfectly known in the sequel.

To support our assumption about the known change profile, it is important to note that the vectors $\theta_1, \ldots, \theta_L$ are defined by the dynamics of equipment and by the type of attack. For some water (or gas) distribution systems, the dynamics of equipment (pumps, compressors, etc.) is *a priori* known or can be pre-calculated (because it is defined by the pump characteristics, pressures, pipeline diameters, volumes of reservoirs, etc.) The *a priori* information on the different types of attack is certainly less reliable. But it can be assumed that each attack scenario leads to a particular attack signature.

Let us consider the following example. It is assumed that the water distribution system is equipped with constant speed pumps (it is a very typical equipment). Hence, a pump has only two operational modes : "on" or "off". If the attacker decides to switch the pump from "off" to "on", then the transient change profile can be calculated from the constructive parameters of pump and from the parameters of the entire system. Hence, there are scenarios in which the transient change "shape" is known, at least approximately. The sensitivity of the proposed algorithm w.r.t. the change profile $\theta_1, \ldots, \theta_L$ will be examined in Subsections 4.4 and 5.3.

3. Detection algorithm

Unfortunately, we cannot establish the (asymptotic) optimality of the proposed VTWL CUSUM test in the class C_α. Some mathematical problems with optimality arising in the case of transient change detection are discussed in Bakhache and Nikiforov (2000). The only available result on the optimality of transient change detection algorithms is obtained for the special case of $L = 1$ for a criterion slightly different from (6) – (7), see Moustakides (2014). It is the Shewhart test, i.e., the repeated Neyman-Pearson test applied to one observation ($L = 1$!). The detection algorithm proposed in this section coincides with the Shewhart test in the case of $L = 1$. Unfortunately, the case of $L = 1$ has a very limited practical application.

On the contrary, Theorem 2 states that the VTWL CUSUM test is optimal for a sub-class of the class C_α by using an upper bound for obtained for $\mathbb{P}_{md}(T_{VTWL})$ in Theorem 1. This sub-class is limited to the repeated one-sided truncated sequential tests (see Theorem 2). Therefore, starting from now, we introduce a sub-optimal algorithm w.r.t. criteria (6) – (7) for detecting the transient change in the stochastic-dynamical system modeled by (11) – (12).

The algorithm is designed in several steps. Firstly, the state space model is reduced to a regression model with redundancy by using a block of last L measurements $y_{k-L+1}, \ldots, y_k$ in Section 3.1. Then, in Section 3.2, the nuisance parameters are eliminated from the observation model by the technique of invariant tests introduced in Fouladirad and Nikiforov (2005). Finally, the VTWL CUSUM test is proposed to detect the transient change in the stochastic-dynamical system with nuisance parameters in Section 3.3.

3.1. Observation model

It is assumed that the change does not occur during the preheating period ($1 \leq k < L$) and that the detection algorithm is not operational during this period. By utilizing a block of last L observations

$y_{k-L+1},\ldots,y_k$, the vector of these observations is rewritten as

$$\underbrace{\begin{bmatrix} y_{k-L+1} \\ y_{k-L+2} \\ \vdots \\ y_k \end{bmatrix}}_{y^k_{k-L+1}} = \underbrace{\begin{bmatrix} C \\ CA \\ \vdots \\ CA^{L-1} \end{bmatrix}}_{\mathscr{C}} x_{k-L+1} + \underbrace{\begin{bmatrix} D & 0 & \cdots & 0 \\ CB & D & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ CA^{L-2}B & CA^{L-3}B & \cdots & D \end{bmatrix}}_{\mathscr{D}} \underbrace{\begin{bmatrix} u_{k-L+1} \\ u_{k-L+2} \\ \vdots \\ u_k \end{bmatrix}}_{u^k_{k-L+1}} +$$
$$\underbrace{\begin{bmatrix} \xi_{k-L+1} \\ \xi_{k-L+2} \\ \vdots \\ \xi_k \end{bmatrix}}_{\xi^k_{k-L+1}} + \underbrace{\begin{bmatrix} G & 0 & \cdots & 0 \\ CF & G & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ CA^{L-2}F & CA^{L-3}F & \cdots & G \end{bmatrix}}_{\mathscr{G}} \underbrace{\begin{bmatrix} d_{k-L+1} \\ d_{k-L+2} \\ \vdots \\ d_k \end{bmatrix}}_{d^k_{k-L+1}} +$$
$$\underbrace{\begin{bmatrix} D_a & 0 & \cdots & 0 \\ CB_a & D_a & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ CA^{L-2}B_a & CA^{L-3}B_a & \cdots & D_a \end{bmatrix}}_{\mathscr{M}} \underbrace{\begin{bmatrix} a_{k-L+1} \\ a_{k-L+2} \\ \vdots \\ a_k \end{bmatrix}}_{\theta^k_{k-L+1}(k_0)}, \tag{13}$$

where $x_{k-L+1} \in \mathbb{R}^n$ is the vector of unknown system states at time $k-L+1$ and $y^k_{k-L+1} \in \mathbb{R}^{Lp}$ is the vector of observations, $u^k_{k-L+1} \in \mathbb{R}^{Lm}$ is the vector of control signals, $d^k_{k-L+1} \in \mathbb{R}^{Lq}$ is the vector of disturbances, $\theta^k_{k-L+1}(k_0) \in \mathbb{R}^{Ls}$ is the vector of transient changes and $\xi^k_{k-L+1} \in \mathbb{R}^{Lp}$ is the vector of sensor noises; the matrices $\mathscr{C} \in \mathbb{R}^{Lp\times n}$, $\mathscr{D} \in \mathbb{R}^{Lp\times Lm}$, $\mathscr{G} \in \mathbb{R}^{Lp\times Lq}$ and $\mathscr{M} \in \mathbb{R}^{Lp\times Ls}$. The observation model can be rewritten in a matrix form as

$$y^k_{k-L+1} = \mathscr{C} x_{k-L+1} + \mathscr{D} u^k_{k-L+1} + \mathscr{G} d^k_{k-L+1} + \mathscr{M}\theta^k_{k-L+1}(k_0) + \xi^k_{k-L+1}. \tag{14}$$

Generally, the control signals u_k are the outputs of the controller then they are known to system operators. In safety-critical infrastructures such as electric power grids, water distribution networks or gas pipelines, the disturbances d_k correspond to the customers' demands which can be estimated by special-designed software. For this reason, let us suppose that the control signals u_k and the disturbances d_k are known. Hence, the vectors $\mathscr{D}u^k_{k-L+1}$ and $\mathscr{G}d^k_{k-L+1}$ are eliminated from (14) by subtraction :

$$r^k_{k-L+1} = y^k_{k-L+1} - \left(\mathscr{D}u^k_{k-L+1} + \mathscr{G}d^k_{k-L+1}\right) = \mathscr{C}x_{k-L+1} + \mathscr{M}\theta^k_{k-L+1}(k_0) + \xi^k_{k-L+1}, \tag{15}$$

where $r^k_{k-L+1} \in \mathbb{R}^{Lp}$ is the vector of simplified observations, which depends on the unknown system states x_{k-L+1}, the vector of transient changes $\theta^k_{k-L+1}(k_0)$ and the vector of sensor noises ξ^k_{k-L+1}. The vector of transient changes $\theta^k_{k-L+1}(k_0)$ depends on the change time k_0 by the following relation:

$$\theta_{k-L+1}^{k}(k_0) = \begin{cases} [0] & \text{if } k < k_0 \\ \begin{bmatrix} [0] \\ \theta_1 \\ \vdots \\ \theta_{k-k_0+1} \end{bmatrix} & \text{if } k_0 \le k < k_0 + L \\ \begin{bmatrix} \theta_{k-k_0-L+2} \\ \vdots \\ \theta_L \\ [0] \end{bmatrix} & \text{if } k_0 + L \le k < k_0 + 2L - 1 \\ [0] & \text{if } k \ge k_0 + 2L - 1 \end{cases}, \tag{16}$$

where $[0]$ is the null vector of appropriate dimension and the attack profile vector $\theta_1, \theta_2, \cdots, \theta_L$ are completely known. Our problem is to detect the presence of transient changes $\theta_{k-L+1}^{k}(k_0)$ based on observation model (15) – (16) and considering the initial state x_{k-L+1} as unknown (and non-random) parameter.

Remark 2. *Let us discuss the vector of transient change $\theta_{k-L+1}^{k}(k_0)$ in (16). During the pre-change mode (i.e., $k < k_0$), it is a null vector. During the period of transient change (i.e., $k_0 \le k \le k_0 + L - 1$), it depends only on the relative position of the change point k_0 inside the time window $[k - L + 1, k]$. In other words, $\theta_{k-L+1}^{k}(k_0) = \theta_1^L(k_0 - k + L)$ for $k - L + 1 \le k_0 \le k$. For example, when $k_0 = k - L + 1$ then $\theta_{k-L+1}^{k}(k - L + 1) = \theta_1^L(1)$ and when $k_0 = k$ then $\theta_{k-L+1}^{k}(k) = \theta_1^L(L)$, for all $k \ge L$. The post-transient change vector $\theta_{k-L+1}^{k}(k_0)$, where $k \ge k_0 + L$, can be potentially used for latent detection mode but it will not be used in this paper since the maximum allowable detection delay is L and all detections with delay greater than L are considered as missed.*

3.2. Nuisance parameter rejection

As it follows from equations (10) – (11), the considered state-space model is only partially stochastic. The process noise is absent and only the measurement (sensor) noise is included. Hence, the state equation is non-random and it is "driven" by the initial state vectors x_0, which is unknown and non-random, and by the known vectors of control signals u_k and disturbances d_k. For the SCADA systems, the known vectors u_k and d_k have a clear significance : the output of controllers and the customers' demands. The impact of these vectors can be easily eliminated (see equation (15)), but the concatenated vector of measurements r_{k-L+1}^{k} also depends on the unknown system state x_{k-L+1}.

The goal of the detection algorithm is to detect the presence of transient change $\theta_{k-L+1}^{k}(k_0)$ completely ignoring the initial state x_{k-L+1}, which does not represent any interest for the transient change detection problem. Unfortunately, the negative impact of the vector x_{k-L+1} on the vector of measurements r_{k-L+1}^{k} represents a serious obstacle. Following the statistical tradition, such parameter x_{k-L+1} is called "nuisance". Hence, to solve the problem of transient change detection, the unknown nuisance parameter x_{k-L+1} has to be rejected from the observations (15) – (16) in order to avoid its negative impact on the VTWL CUSUM test.

The application of invariant hypothesis testing theory to the state space model (10) is discussed in Fouladirad and Nikiforov (2005). The main idea is the following : the observation vector r_{k-L+1}^{k} is projected on the orthogonal complement $R(\mathscr{C})^{\perp}$ of the column space $R(\mathscr{C})$ of matrix $\mathscr{C}$. This projection is the residual vector $\widetilde{r}_{k-L+1}^{k} = \mathscr{W} r_{k-L+1}^{k}$, where the rows of the matrix $\mathscr{W}$ of size $(Lp - \text{rank}(\mathscr{C})) \times Lp$ are composed of the eigenvectors of the projection matrix $\mathscr{P}_{\mathscr{C}} = \mathscr{I} - \mathscr{C}\left(\mathscr{C}^T\mathscr{C}\right)^{-}\mathscr{C}^T$ corresponding to eigenvalue 1, $\left(\mathscr{C}^T\mathscr{C}\right)^{-}$ is the generalized inverse of $\mathscr{C}^T\mathscr{C}$, $\text{rank}(\mathscr{C})$ denotes the rank of $\mathscr{C}$ and $\mathscr{I}$ is the identity matrix of appropriate dimension. The matrix $\mathscr{W}$ satisfies the following conditions $\mathscr{W}\mathscr{C} = 0$, $\mathscr{W}^T\mathscr{W} = \mathscr{P}_{\mathscr{C}}$ and $\mathscr{W}\mathscr{W}^T = \mathscr{I}$. If the matrix $\mathscr{C}$ is of full column rank, then $\mathscr{P}_{\mathscr{C}} = \mathscr{I} - \mathscr{C}\left(\mathscr{C}^T\mathscr{C}\right)^{-1}\mathscr{C}^T$ but this is the worst-case. The smaller is the $\text{rank}(\mathscr{C})$, the less significant is the negative impact of x_{k-L+1}.

Let us assume that the matrix $\mathscr{C}$ is of full column rank n in the rest of the paper. Hence, the residual vector

$$\widetilde{r}_{k-L+1}^{k} = \mathscr{W} r_{k-L+1}^{k} = \mathscr{W}\mathscr{M}\theta_{k-L+1}^{k}(k_0) + \mathscr{W}\xi_{k-L+1}^{k} \tag{17}$$

is independent of the nuisance parameter x_{k-L+1}. It has been shown that the residual vector $\widetilde{r}_{k-L+1}^{k}$ is the maximal invariant statistics (see details in Fouladirad and Nikiforov, 2005). Therefore, any invariant statistics is a function of the maximal invariant $\widetilde{r}_{k-L+1}^{k}$.

Let us investigate now the distribution of the residual vector $\widetilde{r}_{k-L+1}^{k}$. Let $\mathscr{R}$ and Σ be the covariance matrix of the random noise vector ξ_{k-L+1}^{k} and the residual vector $\mathscr{W} r_{k-L+1}^{k}$, respectively. Then,

$$\mathscr{R} = \begin{bmatrix} R & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & R \end{bmatrix} \in \mathbb{R}^{Lp \times Lp}; \quad \Sigma = \mathscr{W}\mathscr{R}\mathscr{W}^T \in \mathbb{R}^{(Lp-n)\times(Lp-n)} \tag{18}$$

since matrix $\mathscr{C}$ is assumed to be full column rank. Hence, the residual vector $\widetilde{r}_{k-L+1}^{k}$ follows the following Gaussian distribution

$$\widetilde{r}_{k-L+1}^{k} \sim \mathscr{N}\left(\mathscr{W}\mathscr{M}\theta_{k-L+1}^{k}(k_0), \Sigma\right), \quad k \geq L, \tag{19}$$

where the parameter vector $\theta_{k-L+1}^{k}(k_0)$ is described in (16). Let us denote by $\widetilde{\mathscr{P}}^{k_0}$ the joint distribution of the random vectors $\widetilde{r}_1^{L}, \widetilde{r}_2^{L+1}, \ldots, \widetilde{r}_{k_0-L+1}^{k_0}, \ldots$ when $\widetilde{r}_{k-L+1}^{k}$ follows the transient change model defined by (16) – (19). This nuisance-free model will be used in the following subsection for designing the detection algorithm.

3.3. *VTWL CUSUM test*

Let us denote the probability measures of the vector $\widetilde{r}_{k-L+1}^{k}$ by $\widetilde{\mathscr{P}}_i$ under the hypothesis of the presence of transient change $\theta_{k-L+1}^{k}(i)$ at position i in the time window $[k-L+1, k]$ (see Remark 2 for details) and by $\widetilde{\mathscr{P}}_0$ under the hypothesis that there is no change in the time window $[k-L+1, k]$. Let us define the VTWL CUSUM test (8) adapted to the state-space model with nuisance parameters. It utilizes the last L observations at each time instant $k \geq L$:

$$T_{VTWL} = \inf\left\{k \geq L : \max_{k-L+1 \leq i \leq k}\left(S_i^k - h_{k-i+1}\right) \geq 0\right\}, \tag{20}$$

where S_i^k is the log-likelihood ratio (LLR) between the probability measures $\widetilde{\mathscr{P}}_i$ and $\widetilde{\mathscr{P}}_0$ written for the vector $\widetilde{r}_{k-L+1}^k$. It is expressed by

$$S_i^k = \log \frac{\widetilde{p}_{\theta_{k-L+1}^k(i)}\left(\widetilde{r}_{k-L+1}^k\right)}{\widetilde{p}_0\left(\widetilde{r}_{k-L+1}^k\right)} = \log \frac{\widetilde{p}_{\theta_{k-L+1}^k(i)}\left(\mathscr{W} r_{k-L+1}^k\right)}{\widetilde{p}_0\left(\mathscr{W} r_{k-L+1}^k\right)}, \tag{21}$$

where $\widetilde{p}_{\theta_{k-L+1}^k(i)}(\cdot)$ (resp. $\widetilde{p}_0(\cdot)$) is the p.d.f. of the probability measure $\widetilde{\mathscr{P}}_i$ (resp. $\widetilde{\mathscr{P}}_0$). In the Gaussian case, the LLR S_i^k is calculated as follows

$$S_i^k = \left[\mathscr{M}\theta_{k-L+1}^k(i)\right]^T \left[\mathscr{W}^T\Sigma^{-1}\mathscr{W}\right]\left[r_{k-L+1}^k - \frac{1}{2}\mathscr{M}\theta_{k-L+1}^k(i)\right], \tag{22}$$

where the vector $\theta_{k-L+1}^k(i)$ is calculated by (16), for $k-L+1 \leq i \leq k$.

The VTWL CUSUM test proceeds as follows. For each time instant $k \geq L$, the VTWL CUSUM test uses a block of L last observations $y_{k-L+1}, \ldots, y_k$ for decision making. First, the simplified observation vector r_{k-L+1}^k is computed by (15). Next, for each time instant i from $k-L+1$ to k, the LLR S_i^k is calculated by (22) and (16). Then, the VTWL CUSUM test compares the LLR S_i^k to the threshold h_{k-i+1} and the alarm T_{VTWL} is declared if one of the LLRs is greater than or equal to its corresponding threshold. The thresholds $h_1, h_2, \cdots, h_L$ are considered as tuning parameters of the VTWL CUSUM test.

4. Statistical performance of the VTWL CUSUM test

The goal of this section is to investigate the statistical performance of the proposed VTWL CUSUM test. First, we show that the worst-case probability of false alarm corresponds to the first time window of size L and we introduce the upper bound on the worst-case probability of missed detection instead of its exact value in Section 4.1. Second, the optimization problem for the VTWL CUSUM test is proposed and solved in Section 4.2. The goal is to minimize the upper bound of the worst-case probability of missed detection $\overline{\mathbb{P}}_{md}$ provided that the worst-case probability of false alarm $\overline{\mathbb{P}}_{fa}$ is upper bounded by a given constant α. It will be shown that the optimized VTWL CUSUM test is equivalent to the FMA test. A numerical method for estimating the worst-case probability of false alarm $\overline{\mathbb{P}}_{fa}$ and the probability of missed detection $\mathbb{P}_{k_0}(T-k_0+1 > L | T \geq k_0)$ is proposed in Section 4.3. Finally, the sensitivity analysis of the FMA test is available in Section 4.4.

Let $\phi_{k-L+1}^k(i) \in \mathbb{R}^{Lp}$, for $k \geq L$ and $k-L+1 \leq i \leq k$, be coefficient vectors calculated from the profile vectors $\theta_{k-L+1}^k(i)$ and system parameters $\mathscr{M}$, $\mathscr{W}$ and Σ as follows:

$$\phi_{k-L+1}^k(i) = \left[\mathscr{W}^T\Sigma^{-1}\mathscr{W}\right]\left[\mathscr{M}\theta_{k-L+1}^k(i)\right], \tag{23}$$

where the coefficient vectors $\phi_{k-L+1}^k(i)$ can be described as (see Remark 3)

$$\phi_{k-L+1}^k(i) = \begin{bmatrix} \phi_1(i) \\ \vdots \\ \phi_L(i) \end{bmatrix}, \; \phi_j(i) \in \mathbb{R}^p, \; k-L+1 \leq i \leq k, \; 1 \leq j \leq L. \tag{24}$$

As shown in Appendix A, the LLR S_i^k, for $k-L+1 \leq i \leq k$ can be represented as a function of the coefficient vectors $\phi_{k-L+1}^k(i)$, the random noise vector ξ_{k-L+1}^k and its mathematical expectation $\mathbb{E}_{k_0}\left[S_i^k\right]$ as follows :

$$S_i^k = \left[\phi_{k-L+1}^k(i)\right]^T \xi_{k-L+1}^k + \mathbb{E}_{k_0}\left[S_i^k\right], \tag{25}$$

where $\mathbb{E}_{k_0}\left[S_i^k\right]$ is the mathematical expectation of S_i^k w.r.t. the distribution $\widetilde{\mathscr{P}}^{k_0}$.

Remark 3. *The coefficient vector $\phi_{k-L+1}^k(i)$, where $k-L+1 \leq i \leq k$, is independent of the time $k \geq L$, i.e., $\phi_{k-L+1}^k(i) = \phi_{m-L+1}^m(i+m-k)$, where $m \geq L$, and it depends only on the relative position of the change point index i inside the time window $[k-L+1,k]$. It will be seen that the coefficient vector $\phi_{k-L+1}^k(i)$ plays a central role in studying the statistical properties of the VTWL CUSUM test.*

4.1. Properties of the error probabilities

Theorem 1 is inspired by Lemma 2.1 and Theorem 2.2, part 1, from Guépié (2013) for the independent observation model (5). In Theorem 1, the results on the probability of false alarm and the probability of missed detection obtained in Guépié (2013) are adapted to the new observation model (15) – (17).

Theorem 1. *Consider the VTWL CUSUM test (20) – (22). Then,*

1. *Let $U_l = \mathbb{P}_0(l \leq T_{VTWL} \leq l+m-1)$ be the probability of false alarm within the time window $[l,l+m-1]$, then $\{U_l\}_{l\geq L}$ is a non-increasing sequence. Hence, the worst-case probability of false alarm within any time window of length m follows*

$$\overline{\mathbb{P}}_{fa}(T_{VTWL};m;h_1,h_2,\cdots,h_L) = \sup_{l\geq L}\mathbb{P}_0(l \leq T_{VTWL} \leq l+m-1) = U_L. \tag{26}$$

2. *The worst-case probability of missed detection is upper bounded by*

$$\overline{\mathbb{P}}_{md}(T_{VTWL};h_1,h_2,\cdots,h_L) \leq \widetilde{\mathbb{P}}_{md}(T_{VTWL};h_L) \triangleq \Phi\left(\frac{h_L-\mu_{S_1^L}}{\sigma_{S_1^L}}\right), \tag{27}$$

where $\Phi(x) = \int_{-\infty}^{x}\frac{1}{\sqrt{2\pi}}\exp\left\{-\frac{1}{2}t^2\right\}dt$ is the c.d.f. of the standard normal distribution, $\widetilde{\mathbb{P}}_{md}(T_{VTWL};h_L)$ is the upper bound on the worst-case probability of missed detection, and $\mu_{S_1^L}$ and $\sigma_{S_1^L}$ are calculated by

$$\mu_{S_1^L} = \frac{1}{2}\left[\mathscr{M}\theta_1^L(1)\right]^T\left[\mathscr{W}^T\Sigma^{-1}\mathscr{W}\right]\left[\mathscr{M}\theta_1^L(1)\right], \tag{28}$$

$$\sigma_{S_1^L}^2 = \left[\mathscr{M}\theta_1^L(1)\right]^T\left[\mathscr{W}^T\Sigma^{-1}\mathscr{W}\right]\left[\mathscr{M}\theta_1^L(1)\right]. \tag{29}$$

Proof. The proof of Theorem 1 is given in Appendix B. □

Remark 4. *Let us discuss the results of Theorem 1. First, it has been shown that the worst-case probability of false alarm $\overline{\mathbb{P}}_{fa}(T_{VTWL};m;h_1,h_2,\cdots,h_L)$ corresponds to the time window $[L,L+m-1]$. This information is necessary to proceed the optimization problem for selecting* *the thresholds of the VTWL CUSUM test. Second, it has been shown that the upper bound* $\widetilde{\mathbb{P}}_{md}(T_{VTWL};h_L)$ *for the probability of missed detection is a function of only* h_L. *It will be seen in Theorem 2 that this fact is also very useful to solve the problem of the VTWL CUSUM test optimization.*

4.2. Optimization of the VTWL CUSUM test

To optimize the VTWL CUSUM test it is necessary to minimize the worst-case probability of missed detection $\overline{\mathbb{P}}_{md}(T_{VTWL};h_1,h_2,\cdots,h_L)$ provided that the worst-case probability of false alarm $\overline{\mathbb{P}}_{fa}(T_{VTWL};m;h_1,h_2,\cdots,h_L)$ is upper bounded by α. It appears that the exact expression of $\overline{\mathbb{P}}_{md}(T_{VTWL};h_1,h_2,\cdots,h_L)$ is complicated due to some mathematical difficulties. For this reason, it is proposed to minimize its upper bound $\widetilde{\mathbb{P}}_{md}(T_{VTWL};h_L)$ given by (27) instead of the worst-case probability of missed detection. An empirical evaluation of the upper bound sharpness can be done by using the comparison between the exact value of missed detection probability and its upper bound $\widetilde{\mathbb{P}}_{md}(T_{FMA};h_L)$ proposed in Section 5.2.

Let us impose the following constraints on the coefficient vector $\phi_1^L(1)$:

Assumption 1. *As it follows from* (23) *and Remark 3, it is sufficient to define the properties of the coefficient vectors* $\phi_{k-L+1}^k(i)$ *for* $k=L$. *Let us consider the coefficient vector* $\phi_1^L(1)=\left[\phi_1^T(1)\cdots\phi_L^T(1)\right]^T$. *It is assumed that there exists at least one vector* $\phi_j^T(1)\neq 0$ *for* $1\leq j\leq L$.

Remark 5. *Sometimes, the change profile* $\theta_{k-L+1}^k(i)$ *can be undetectable due to the mutual properties of the matrices* $\mathscr{C}$ *and* $\mathscr{M}$, *see* (14). *If the subspace spanned by the columns of the matrix* $\mathscr{M}$ *belongs to the subspace spanned by the columns of the matrix* $\mathscr{C}$, *then the product* $\mathscr{W}\mathscr{M}$ *is equal to zero (see details in Fouladirad and Nikiforov, 2005). Such undetectable transient changes are out of the scope of this paper. Some additional discussion on the detectability in the case of nuisance parameters can be found in Fouladirad and Nikiforov (2005); Fillatre and Nikiforov (2007).*

Lemma 1. *Let* $\mathscr{S}\in\mathbb{R}^m$ *be a Gaussian random vector consisting of m LLRs* $S_1^L,S_2^{L+1},\cdots,S_m^{L+m-1}$. *If Assumption 1 is satisfied, then the covariance matrix* $\Sigma_{\mathscr{S}}\in\mathbb{R}^{m\times m}$ *of the Gaussian random vector* $\mathscr{S}$ *is positive definite.*

Proof. The proof of Lemma 1 is given in Appendix C. □

The positive definiteness of the covariance matrix $\Sigma_{\mathscr{S}}\in\mathbb{R}^{m\times m}$ is used in the following theorem.

Theorem 2. *Consider the VTWL CUSUM test* (20) – (22). *Then,*

1. *The optimal choice of the thresholds* $h_1,h_2,\cdots,h_L$ *leads to the following optimization problem:*

$$\begin{cases}\inf_{h_1,h_2,\cdots,h_L} & \widetilde{\mathbb{P}}_{md}(T_{VTWL};h_L)\\ \text{subject to} & \overline{\mathbb{P}}_{fa}(T_{VTWL};m;h_1,h_2,\cdots,h_L)\leq\alpha\end{cases}, \tag{30}$$

where α *is the acceptable level for the worst-case probability of false alarm within any time window of length m. The optimization problem* (30) *has the unique solution* $(h_1^*,h_2^*,\cdots,h_L^*)$

for a given $\alpha \in (0,1)$, where $h_1^*, h_2^*, \cdots, h_{L-1}^* \to +\infty$ and h_L^* is calculated from the following equation:

$$\mathbb{P}_0\left(\bigcap_{k=L}^{L+m-1}\left\{S_{k-L+1}^k < h_L^*\right\}\right) = 1-\alpha. \tag{31}$$

2. *The optimized VTWL CUSUM test is equivalent to the following FMA test:*

$$T_{FMA}\left(\widetilde{h}_L\right) = \inf\left\{k \geq L : \left[\mathscr{M}\theta_1^L(1)\right]^T \left[\mathscr{W}^T\Sigma^{-1}\mathscr{W}\right] r_{k-L+1}^k \geq \widetilde{h}_L\right\}, \tag{32}$$

where the threshold $\widetilde{h}_L$ of the FMA test is calculated from the optimal threshold h_L^* of the VTWL CUSUM test by

$$\widetilde{h}_L = h_L^* + \mu_{S_1^L}. \tag{33}$$

The upper bound on the worst-case probability of missed detection of the FMA test as a function of the threshold $\widetilde{h}_L$ is given by

$$\overline{\mathbb{P}}_{md}\left(T_{FMA};\widetilde{h}_L\right) \leq \widetilde{\mathbb{P}}_{md}\left(T_{FMA};\widetilde{h}_L\right) \triangleq \Phi\left(\frac{\widetilde{h}_L - 2\mu_{S_1^L}}{\sigma_{S_1^L}}\right). \tag{34}$$

Proof. The proof of Theorem 2 is given in Appendix D. □

4.3. Numerical computation of the error probabilities

A numerical method for estimating the worst-case probability of false alarm and the probability of missed detection for the general VTWL CUSUM test and for a particular case, i.e., the FMA test, is proposed. The obtained results are also applicable to the the WL CUSUM test which is another particular case of the VTWL CUSUM test. It will be necessary for the comparison between the WL CUSUM and FMA tests in Section 5.

Proposition 1. *The worst-case probability of false alarm and the probability of missed detection for the VTWL CUSUM test in (20) – (22) and the FMA detection rule in (32) are calculated numerically by the following formulas:*

1. *The worst-case probability of false alarm is computed as*

$$\overline{\mathbb{P}}_{fa}\left(T_{VTWL};m;h_1,h_2,\cdots,h_L\right) = 1-\mathbb{P}_0\left(\bigcap_{k=L}^{L+m-1}\bigcap_{i=k-L+1}^{k}\left\{S_i^k < h_{k-i+1}\right\}\right), \tag{35}$$

$$\overline{\mathbb{P}}_{fa}\left(T_{FMA};m;\widetilde{h}_L\right) = 1-\mathbb{P}_0\left(\bigcap_{k=L}^{L+m-1}\left\{S_{k-L+1}^k < \widetilde{h}_L - \mu_{S_1^L}\right\}\right). \tag{36}$$

2. *The conditional probability of missed detection is calculated as a function of k_0 is given as*

follows

$$\mathbb{P}_{k_0}\left(T_{VTWL} \geq k_0 + L | T_{VTWL} \geq k_0\right) = \frac{\mathbb{P}_{k_0}\left(\bigcap_{k=L}^{k_0+L-1} \bigcap_{i=k-L+1}^{k} \left\{S_i^k < h_{k-i+1}\right\}\right)}{\mathbb{P}_{k_0}\left(\bigcap_{k=L}^{k_0-1} \bigcap_{i=k-L+1}^{k} \left\{S_i^k < h_{k-i+1}\right\}\right)}, \quad (37)$$

$$\mathbb{P}_{k_0}\left(T_{FMA} \geq k_0 + L | T_{FMA} \geq k_0\right) = \frac{\mathbb{P}_{k_0}\left(\bigcap_{k=L}^{k_0+L-1} \left\{S_{k-L+1}^k < \widetilde{h}_L - \mu_{S_1^L}\right\}\right)}{\mathbb{P}_{k_0}\left(\bigcap_{k=L}^{k_0-1} \left\{S_{k-L+1}^k < \widetilde{h}_L - \mu_{S_1^L}\right\}\right)}. \quad (38)$$

Proof. The proof of equations (35) – (38) is given in Appendix E. □

The numerical realisation of Proposition 1 is based on the calculation of the multivariate normal c.d.f. introduced in Genz and Bretz (2002). This algorithm has been implemented in Matlab's Statistics Toolbox by the function mvncdf.

4.4. Sensitivity analysis of the FMA test

The goal of this subsection is to analyze the sensitivity of the FMA test w.r.t. the operational parameters, including the attack duration, the attack profiles, and the sensor noise covariance matrix. Since the operational parameters are rarely exactly known, it is important for practical applications.

Let L and $\overline{L}$ be, respectively, the putative and true values of the attack duration. Let also $\theta_1, \theta_2, \cdots, \theta_L$ and $\overline{\theta}_1, \overline{\theta}_2, \cdots, \overline{\theta}_L$ denote the putative and true values of the attack profiles, respectively. Finally, let R and $\overline{R}$ stand for the putative and true values of the sensor noise covariance matrix, respectively. It is worth noting that the putative operational parameters (i.e., L, $\theta_1, \theta_2, \cdots, \theta_L$, and R) remain unchanged and they are considered as the designed parameters. The variation of true values (i.e., $\overline{L}$, $\overline{\theta}_1, \overline{\theta}_2, \cdots, \overline{\theta}_L$, and $\overline{R}$) leads to some change in parameters of the statistical model. However, the proposed numerical method can also be used to analyze the sensitivity of the FMA test w.r.t. these parameters.

The formulas for estimating the worst-case probability of false alarm $\overline{\mathbb{P}}_{fa}\left(T_{FMA}; m; \widetilde{h}_L\right)$ and the probability of missed detection $\overline{\mathbb{P}}_{md}\left(T_{FMA}; \widetilde{h}_L\right)$ remain unchanged. It is necessary to modify the mathematical expectations $\mathbb{E}_0\left[S_i^k\right]$ and $\mathbb{E}_{k_0}\left[S_i^k\right]$ and the covariance $\operatorname{cov}\left(S_{i_1}^{k_1}, S_{i_2}^{k_2}\right)$.

As it follows from Appendix A.1 (see equations (44) – (46)), the mathematical expectation $\mathbb{E}_0\left[S_i^k\right]$, given by (46), remains unchanged and the modified mathematical expectation $\mathbb{E}_{k_0}\left[S_i^k\right]$ is given by the following expression

$$\mathbb{E}_{k_0}\left[S_i^k\right] = \left[\mathscr{M}\theta_{k-L+1}^k(i)\right]\left[\mathscr{W}^T\Sigma^{-1}\mathscr{W}\right]\left[\mathscr{M}\overline{\theta}_{k-L+1}^k(k_0) - \frac{1}{2}\mathscr{M}\theta_{k-L+1}^k(i)\right], \quad (39)$$

where $\overline{\theta}_{k-L+1}^{k}(k_0)$ is the vector of true transient profiles, formulated in the same manner as $\theta_{k-L+1}^{k}(k_0)$ by replacing the putative attack profiles $\theta_1, \theta_2, \cdots, \theta_L$ by the true attack profiles $\overline{\theta}_1, \overline{\theta}_2, \cdots, \overline{\theta}_L$.

The calculation of $\operatorname{cov}\left(S_{i_1}^{k_1}, S_{i_2}^{k_2}\right)$ is performed in exactly the same way as it has been done in Appendix A.2. The modifications are included in the final formula. Hence, the modified covariance $\operatorname{cov}\left(S_{i_1}^{k_1}, S_{i_2}^{k_2}\right)$ is given by

$$\operatorname{cov}\left(S_{i_1}^{k_1}, S_{i_2}^{k_2}\right) = \sum_{t_0=i_{max}}^{k_{min}} \left[\phi_{t_0-k_1+L}^{T}(i_1)\,\overline{R}\,\phi_{t_0-k_2+L}(i_2)\right], \tag{40}$$

where the putative covariance matrix R in equation (52) is replaced by its true value $\overline{R}$.

The application of these modified equations (39) – (40) to a simple SCADA system monitoring and to numerical examples will be discussed in the following section.

5. Application and numerical examples

Nowadays, the majority of safety-critical infrastructures, for example nuclear facilities, electric power grids, drinking water distribution networks or gas pipelines, are controlled and monitored by the SCADA systems. Along with the development in communication technology, these communication-based systems become more and more vulnerable to cyber/physical attacks. Needless to say that the greater concern should be paid for ensuring the security of SCADA systems so as to avoid physical destruction, economic losses, or even human life.

The problem of detecting and identifying cyber/physical attacks on networked control systems has attracted increasing attention from research community, see Amin et al. (2012a,b); Pasqualetti et al. (2013); Cárdenas et al. (2011), especially after the Stuxnet virus incident Brunner et al. (2010). Generally, the attack detection problem is transformed into the problem of detecting abrupt changes in both state evolution and sensor measurement equations.

In this section, we utilize the proposed algorithms for detecting cyber/physical attacks on a simple SCADA water distribution system. First, we introduce the architecture and the model of the water distribution system under normal operation as well as under cyber/physical attacks in Section 5.1. We consider the scenario where the attack is designed for stealing water from the reservoir, turning off the pump and compromising the sensor measurements. Next, the WL CUSUM and FMA tests are applied to detect the attack in Section 5.2. Their statistical properties are investigated and compared by using the proposed numerical method and the Monte Carlo simulation also in Section 5.2. Finally, the numerical analysis of the FMA test sensitivity is given in Section 5.3.

5.1. SCADA water distribution system

Let us consider a simple SCADA water distribution system which is shown in Figure 1. The system is composed of a treatment plant W_1, a reservoir R_1, a pump P_1, 3 junctions N_2, N_3 and N_4, 4 pipelines C_{01}, C_{12}, C_{23}, and C_{24} and 2 consumers d_1 and d_2. Two pressure sensors S_1 and S_2 are equipped for measuring pressure heads h_1 at the reservoir and h_2 at the node N_2, respectively.

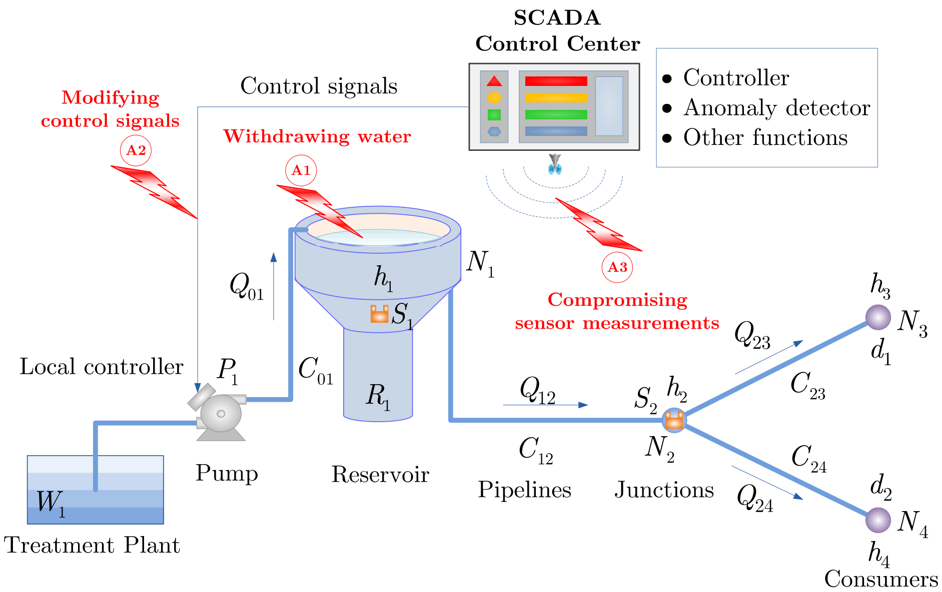

FIGURE 1. *A simple SCADA water distribution system : the case study.*

A linearized model of a water distribution network is obtained by using the mass balance equation and the energy balance equation (see the detailed description in Pasqualetti, 2012, pages 35–36). The system can be described by the discrete-time state space model

$$\begin{cases} x_{k+1} & = Ax_k + Bu_k + Fd_k \\ y_k & = Cx_k + Du_k + Gd_k \end{cases} ; \quad x_0 = \bar{x}_0, \tag{41}$$

where $x_k \in \mathbb{R}$ is the pressure head h_1 at the reservoir with initial value $\bar{x}_0$, $u_k \in \mathbb{R}$ is the control signal sent from the control center to the local controller which regulates the flow rate Q_{01} through the pump (supplying water to the reservoir), $d_k \in \mathbb{R}^2$ is the disturbances which correspond to the consummation of customers at nodes N_3 and N_4, $y_k \in \mathbb{R}^2$ is the measurements of sensors S_1 and S_2 with the sensor noises $\xi_k \sim \mathcal{N}(0,R)$; the matrices $A \in \mathbb{R}^{1\times 1}$, $B \in \mathbb{R}^{1\times 1}$, $F \in \mathbb{R}^{1\times 2}$, $C \in \mathbb{R}^{2\times 1}$, $D \in \mathbb{R}^{2\times 1}$, $G \in \mathbb{R}^{2\times 2}$ and $R \in \mathbb{R}^{2\times 2}$.

Let us consider the scenario of a coordinated attack, where the attacker simultaneously performs three actions : 1) stealing water from the reservoir with a constant flow rate Q_0; 2) turning off the pump and 3) compromising the measurements of sensors S_1 and S_2 during the attack period $\tau_a = [k_0, k_0 + L - 1]$, where the attack instant k_0 is unknown (for the attack detection algorithm). This attack scenario is motivated by a real attack on city water utility where the pump was burned out after being turned on and off, as reported in Zetter (2011). The system model under attack is described as

$$\begin{cases} x_{k+1} & = Ax_k + Bu_k + Fd_k + B_a a_k \\ y_k & = Cx_k + Du_k + Gd_k + D_a a_k \end{cases} ; \quad x_0 = \bar{x}_0, \tag{42}$$

where the attack vector $a_k \in \mathbb{R}^4$ is designed by the adversary and the matrices $B_a \in \mathbb{R}^{1\times 4}$ and $D_a \in \mathbb{R}^{2\times 4}$.

5.2. Numerical results for exactly known parameters

The goal of this subsection is to use the above mentioned example to compare the proposed numerical method of the error probabilities computation for the FMA and WL CUSUM tests against the results of Monte Carlo simulation.

Remark 6. *The only way to verify our numerical algorithm, in the absence of analytical expressions for the probabilities of errors, is to compare the probabilities of errors with the results of a 10^6-repetition Monte Carlo simulation, which is very time consuming but can be realized for a limited number of numerical experiments. Obviously, the proposed numerical method dramatically reduces the computational time w.r.t. the Monte Carlo simulation. It is also worth noting that the computational time for the calculation of $\overline{\mathbb{P}}_{fa}(T_{FMA})$ and $\overline{\mathbb{P}}_{md}(T_{FMA})$ for the FMA test by the proposed numerical method is much smaller than the computational time of the general VTWL CUSUM test.*

The system parameters are chosen as follows. The sampling period $T_S = 100$ s and the initial pressure head $\bar{x}_0 = 100$ m. The system matrices $A = 1$, $B = 0.5$, $F = \begin{bmatrix} -0.5 & -0.5 \end{bmatrix}$, $C = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $D = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, $G = \begin{bmatrix} 0 & 0 \\ -10 & -10 \end{bmatrix}$, $R = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $B_a = \begin{bmatrix} 0.5 & 0.5 & 0 & 0 \end{bmatrix}$ and $D_a = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$. The parameters of the statistical model $\mathscr{C}$, $\mathscr{D}$, $\mathscr{G}$, $\mathscr{W}$, $\mathscr{R}$, Σ can be calculated from the system parameters A, B, F, C, D, G, B_a, D_a and R. Without loss of generality, it is assumed that $u_k = u_0 = 1$ for supplying the reservoir with $Q_{01} = 1$ m^3/s and the customers' demands fluctuate around the value $d_{1,k} \approx d_{2,k} \approx 0.5$ m^3/s.

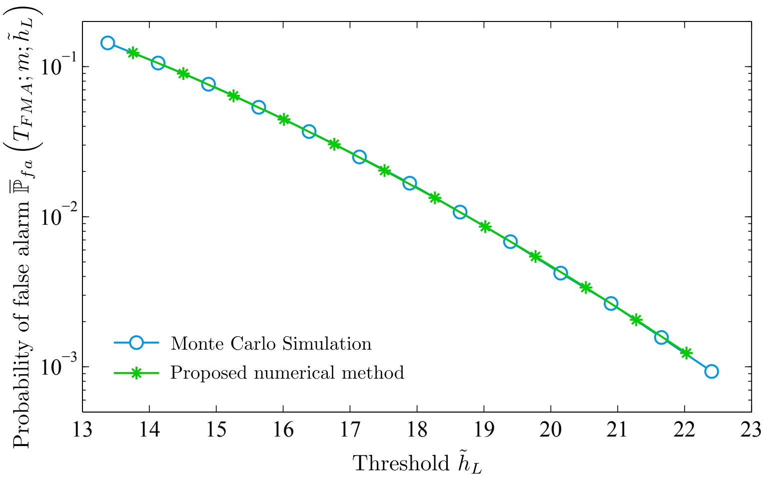

FIGURE 2. *The worst-case probability of false alarm $\overline{\mathbb{P}}_{fa}\left(T_{FMA};m;\widetilde{h}_L\right)$ of the FMA test as a function of the threshold $\widetilde{h}_L$, calculated by 10^6-repetition Monte Carlo simulation and by the numerical method.*

The attack parameters are chosen as follows. The stolen flow rate is $Q_0 = 0.2$ m^3/s. The attack duration $L = 8$ observations, corresponding to a period of 13.3 min. The false alarm rate is measured by the time window of length $m = 3L = 24$ observations, being equivalent to a duration

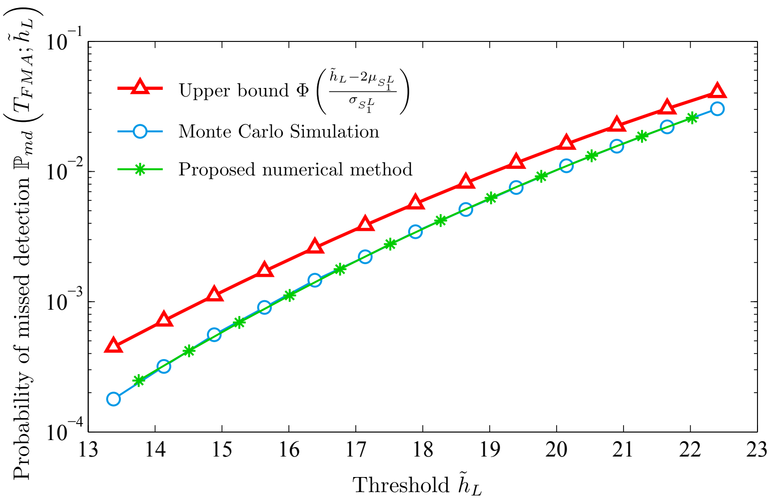

FIGURE 3. *The probability of missed detection $\overline{\mathbb{P}}_{md}\left(T_{FMA};\widetilde{h}_L\right)$ of the FMA detection rule, calculated by a 10^6-repetition Monte Carlo simulation and by the numerical method and its upper bound $\Phi\left[(\widetilde{h}_L - 2\mu_{S_1^L})/\sigma_{S_1^L}\right]$ as functions of the threshold $\widetilde{h}_L$.*

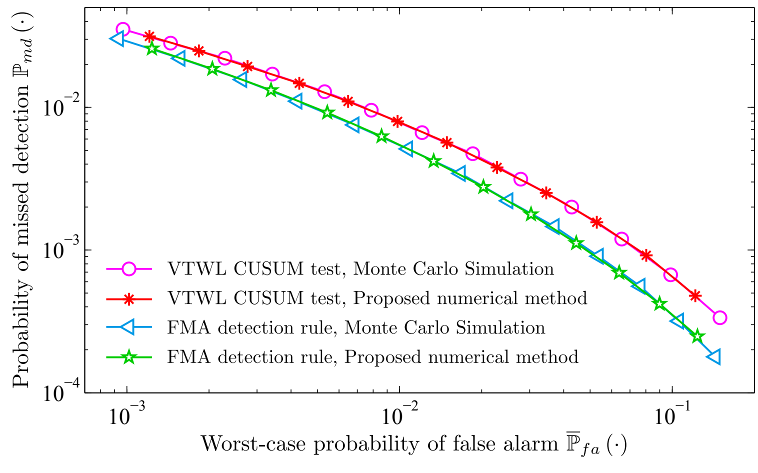

FIGURE 4. *Comparison between the FMA and WL CUSUM tests. The probability of missed detection against the worst-case probability of false alarm calculated by 10^6-repetition Monte Carlo simulation and by the numerical method.*

of 40 min. The attack vector a_k can be designed by the covert attack strategy, which was first introduced in Smith (2011), as follows:

$$a_k = \begin{cases} [0] & \text{if } k < k_0 \\ \begin{bmatrix} -0.2 \\ -1 \\ 0.6\,(k-k_0) \\ 0.6\,(k-k_0) \end{bmatrix} & \text{if } k_0 \le k < k_0 + L\,. \\ [0] & \text{if } k \ge k_0 + L \end{cases} \tag{43}$$

where $[0] = (0,\dots,0)^T$ is the null vector of appropriate dimension.

The attack vector $a_k \in \mathbb{R}^4$ contains all information about the attack. The first element reflects the physical attack to withdraw water from the reservoir with the flow rate $Q_0 = 0.2\ {}^{\mathrm{m}^3}\!/\mathrm{s}$. The second element reflects the cyber attack on the control signal for turning off the pump. The modification of the sensor measurements is reflected by the two last elements.

The worst-case probability of false alarm $\overline{\mathbb{P}}_{fa}\left(T_{FMA}; m; \widetilde{h}_L\right)$, calculated by the proposed numerical method, is compared against a 10^6-repetition Monte Carlo simulation of the FMA test. The results for both methods as functions of $\widetilde{h}_L : \widetilde{h}_L \mapsto \overline{\mathbb{P}}_{fa}\left(T_{FMA}; m; \widetilde{h}_L\right)$ are shown in Figure 2. As it follows from Figure 2, two curves (numerical and Monte Carlo) perfectly coincide, thus confirming the precision of the proposed numerical method.

The probability of missed detection $\overline{\mathbb{P}}_{md}\left(T_{FMA}; \widetilde{h}_L\right)$, calculated twice, by the proposed numerical method and by a 10^6-repetition Monte Carlo simulation, is compared against its upper bound $\widetilde{\mathbb{P}}_{md}\left(T_{FMA}; \widetilde{h}_L\right)$ given by (34). Three curves (upper bound, numerical and Monte Carlo) are shown in Figure 3. The change point has been chosen as $k_0 = L+1$. As it follows from Figure 3, the proposed upper bound is relatively sharp. Again, the numerical and Monte Carlo curves perfectly coincide, thus confirming the precision of the numerical method.

The conventional CUSUM test (optimal for the classical non-Bayesian change detection) has been also proposed and studied for transient change detection (see for example Bakhache and Nikiforov, 2000; Han et al., 1999). Motivated by this fact, it has been shown in Guépié et al. (2012a), by Monte Carlo simulation, that the FMA test performs much better than the conventional CUSUM test (see Figure 4 in Guépié et al., 2012a). Let us now compare the FMA test against the WL CUSUM test, which is another popular and asymptotically optimal test for the classical non-Bayesian change detection, introduced in Willsky and Jones (1976) and studied in Lai (1995, 1998). It follows from (8) that the WL CUSUM test is a particular case of the VTWL CUSUM test, where the thresholds are selected as $h_1 = h_2 = \cdots = h_L$ (see details in Lai, 1995, 1998. The results of comparison are shown in Figure 4, where the probability of missed detection is represented as a function of the worst-case probability of false alarm. The change time is chosen as $k_0 = L+1$. It can be concluded from the Figure 4 that the FMA test also outperforms the WL CUSUM test. Again, the numerical and Monte Carlo curves perfectly coincide, thus confirming the precision of the proposed numerical method.

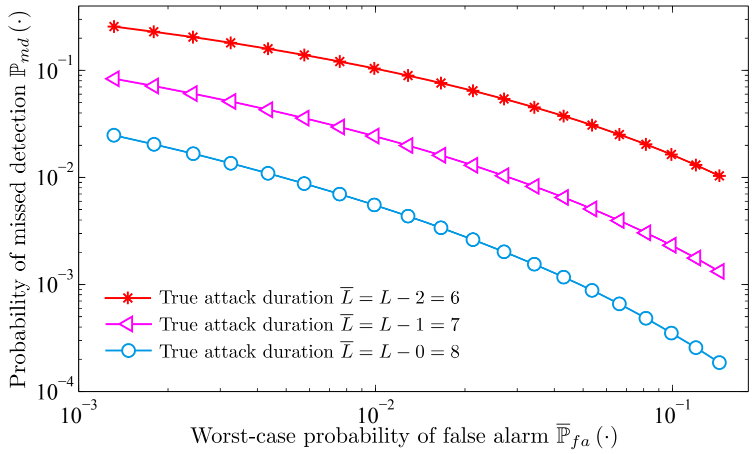

FIGURE 5. *The probability of missed detection $\overline{\mathbb{P}}_{md}(\cdot)$ as a function of the worst-case probability of false alarm $\overline{\mathbb{P}}_{fa}(\cdot)$ calculated for different values of the true attack duration $\overline{L} = \{6,7,8\}$ by the numerical method.*

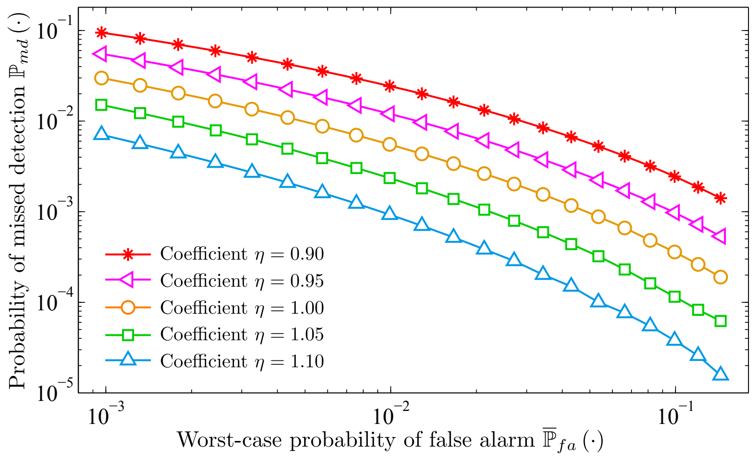

FIGURE 6. *The probability of missed detection $\overline{\mathbb{P}}_{md}(\cdot)$ as a function of the worst-case probability of false alarm $\overline{\mathbb{P}}_{fa}(\cdot)$ calculated for different values of $\eta = \{0.90, 0.95, 1.00, 1.05, 1.10\}$ by the numerical method. The true attack profile is chosen as $\overline{\theta}_j = \eta\theta_j$ for $1 \leq j \leq L$.*

5.3. Numerical results for the sensitivity analysis

In Section 4.4, we have proposed a numerical method for evaluating the sensitivity of the FMA test w.r.t. several operational parameters, including the attack duration, the attack profiles and the sensor noise covariance matrix. In the following, the results of Section 4.4 are applied to the above defined numerical example.

The sensitivity of the FMA test w.r.t. the attack duration is shown in Figure 5, where the putative attack duration is chosen as $L = 8$ and the its true value is $\overline{L} = \{6,7,8\} \leq L$. If the true attack duration is greater than the putative value (i.e., $\overline{L} > L$), the probability of missed detection $\overline{\mathbb{P}}_{md}$ remains unchanged since any detection with the detection delay greater than L is considered as missed. For $\overline{L} = \{6,7,8\} \leq L$, the probability of missed detection $\overline{\mathbb{P}}_{md}$ depends heavily on the true attack duration $\overline{L}$. The smaller the true attack duration $\overline{L}$, the higher the probability of missed detection $\overline{\mathbb{P}}_{md}$. This phenomenon can be explained by the fact that small attack duration $\overline{L}$ causes little changes in the distribution of the observations, thus increasing the probability of missed detection $\overline{\mathbb{P}}_{md}$. It is clear that the probability of false alarm $\overline{\mathbb{P}}_{fa}$ remains unchanged since all the observations are generated from $\widetilde{\mathscr{P}}_0$. Therefore, the interpretation of Figure 5 is very simple: each value of the probability of false alarm $\overline{\mathbb{P}}_{fa}$ corresponds to a certain value of the threshold $\widetilde{h}_L$ (see Figure 2), which is a tuning parameter of the FMA test. Hence, by drawing a vertical line, someone can estimate the variation of the probability of missed detection $\overline{\mathbb{P}}_{md}$ due to the true attack duration smaller than its putative value for a given tuning of the FMA test.

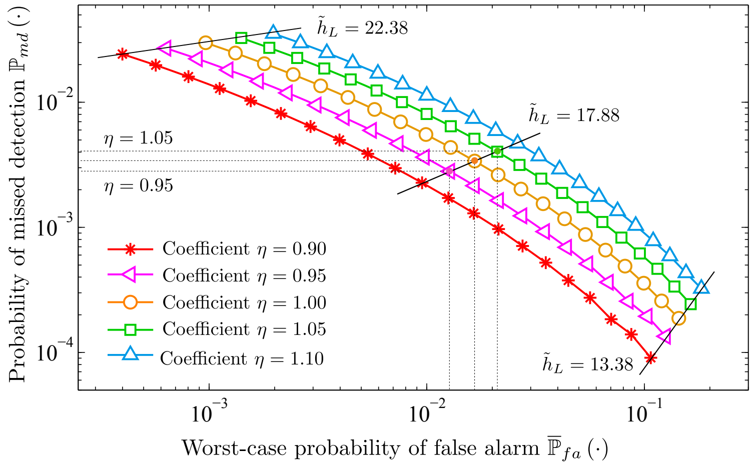

FIGURE 7. *The probability of missed detection $\overline{\mathbb{P}}_{md}(\cdot)$ as a function of the worst-case probability of false alarm $\overline{\mathbb{P}}_{fa}(\cdot)$ calculated for different values of $\eta = \{0.90, 0.95, 1.00, 1.05, 1.10\}$ by the numerical method. The true sensor covariance matrix is chosen as $\overline{R} = \eta R$.*

The sensitivity of the FMA test w.r.t. the attack profile is shown in Figure 6. The putative attack profile is given by $\theta_1, \theta_2, \cdots, \theta_L$ and the true attack profile $\overline{\theta}_1, \overline{\theta}_2, \cdots, \overline{\theta}_L$ is chosen such as $\overline{\theta}_j = \eta \theta_j$ for $1 \leq j \leq L$, where $\eta = \{0.90, 0.95, 1.00, 1.05, 1.10\}$. In other words, the "magnitude" of the change varies from 90% to 110% but the "shape" of the change remains constant. Similar to the attack duration case, the probability of false alarm $\overline{\mathbb{P}}_{fa}$ is independent of the true attack profiles $\overline{\theta}_1, \overline{\theta}_2, \cdots, \overline{\theta}_L$. In contrast, the probability of missed detection $\overline{\mathbb{P}}_{md}$ depends heavily on the true attack profiles $\overline{\theta}_1, \overline{\theta}_2, \cdots, \overline{\theta}_L$. The higher the true attack profiles $\overline{\theta}_1, \overline{\theta}_2, \cdots, \overline{\theta}_L$, the smaller the probability of missed detection $\overline{\mathbb{P}}_{md}$. The interpretation of Figure 6 w.r.t. the tuning parameter $\widetilde{h}_L$ is exactly the same as in the previous case.

Finally, the sensitivity of the FMA test w.r.t. the sensor noises is shown in Figure 7. The putative value of sensor noise covariance matrix is R and its true value $\overline{R}$ is such chosen as $\overline{R} = \eta R$, where $\eta = \{0.90, 0.95, 1.00, 1.05, 1.10\}$. Here, the difference $\overline{R} - R$ impacts both, the probability of false alarm $\overline{\mathbb{P}}_{fa}$ and the probability of missed detection $\overline{\mathbb{P}}_{md}$. Roughly speaking, the bigger the sensor noises, the higher the error probabilities, i.e., $\overline{\mathbb{P}}_{fa}$ and $\overline{\mathbb{P}}_{md}$. For this reason, the interpretation of Figure 7 w.r.t. the threshold (tuning parameter) $\widetilde{h}_L$ is more complicated. To simplify the interpretation of Figure 7, three isolines of constant threshold $\widetilde{h}_L$ are shown in Figure 7. Therefore, by choosing the curve corresponding to $\eta = 1$ and by using Figure 2, someone fixes the tuning value of threshold $\widetilde{h}_L$. Next, by using the isoline intersecting the curve corresponding to $\eta = 1$, someone estimates the variations of error probabilities $\overline{\mathbb{P}}_{fa}$ and $\overline{\mathbb{P}}_{md}$ due to the true sensor noise covariance matrix $\overline{R}$ different from its putative value. This situation is illustrated in Figure 7 for the isoline of $\widetilde{h}_L = 17.88$ by vertical and horizontal dotted lines showing the variation of the error probabilities, i.e., $\overline{\mathbb{P}}_{fa}$ and $\overline{\mathbb{P}}_{md}$. This analysis could help in finding a tradeoff between the performance of the detection algorithms and the price of high-precision sensors.

6. Conclusion

The sequential detection of transient changes in stochastic-dynamical systems with nuisance parameters has been addressed in the paper. To eliminate the nuisance parameters from the observation model, the maximal invariant statistics is used. The Variable Threshold Window Limited CUmulative SUM (VTWL CUSUM) test has been designed. In contrast to the previously published case of independent observations Guépié et al. (2012a,b); Guépié (2013), the considered stochastic-dynamical systems with nuisance parameters do not allow us to assume that $\text{cov}\left(S_{i_1}^{k_1}, S_{i_2}^{k_2}\right) \geq 0$. For this reason, an analytical expression for the worst-case probability of false alarm has been replaced by a numerical method for computing the probability of false alarm and the probability of missed detection. It has been shown that the VTWL CUSUM test optimization w.r.t. the optimality criterion (6) – (7) leads to the Finite Moving Average (FMA) test. The proposed numerical method has been applied to study and to compare the statistical properties of the VTWL CUSUM and FMA tests. The theoretical results have been verified by Monte Carlo simulation in the context of detecting cyber/physical attacks on a simple SCADA water distribution system, targeting at disrupting the system operation.

Appendix A: Calculation of $\mathbb{E}_{k_0}\left[S_i^k\right]$, $\mathbb{E}_0\left[S_i^k\right]$ and $\text{cov}\left(S_{i_1}^{k_1}, S_{i_2}^{k_2}\right)$

Before investigating the statistical performance of the VTWL CUSUM test, let us calculate the mathematical expectation of the Gaussian random variable S_i^k for any positive integers $k \geq L$ and $k-L+1 \leq i \leq k$ and the covariance between two Gaussian random variables $S_{i_1}^{k_1}$ and $S_{i_2}^{k_2}$, for any positive integers i_1, k_1, i_2, k_2 satisfying $k_1, k_2 \geq L$ and $k_1 - L + 1 \leq i_1 \leq k_1$, $k_2 - L + 1 \leq i_2 \leq k_2$.

A.1. Calculation of $\mathbb{E}_{k_0}\left[S_i^k\right]$ *and* $\mathbb{E}_0\left[S_i^k\right]$

It follows from (15), (17) and (22) that

$$S_i^k = \left[\mathscr{M}\theta_{k-L+1}^k(i)\right]^T \left[\mathscr{W}^T\Sigma^{-1}\mathscr{W}\right] \left[\mathscr{M}\theta_{k-L+1}^k(k_0) - \frac{1}{2}\mathscr{M}\theta_{k-L+1}^k(i) + \xi_{k-L+1}^k\right], \tag{44}$$

where the random noise vector $\xi_{k-L+1}^k \sim \mathscr{N}(0, \mathscr{R})$ and the matrices $\mathscr{M}$, $\mathscr{W}$, $\mathscr{R}$, Σ are calculated from system parameters, leading to

$$\mathbb{E}_{k_0}\left[S_i^k\right] = \left[\mathscr{M}\theta_{k-L+1}^k(i)\right]^T \left[\mathscr{W}^T\Sigma^{-1}\mathscr{W}\right] \left[\mathscr{M}\theta_{k-L+1}^k(k_0) - \frac{1}{2}\mathscr{M}\theta_{k-L+1}^k(i)\right], \tag{45}$$

where $\mathbb{E}_{k_0}\left[S_i^k\right]$ is the mathematical expectation of S_i^k w.r.t. the distribution $\widetilde{\mathscr{P}}^{k_0}$. Let us consider the pre-change measure $\widetilde{\mathscr{P}}_0 = \widetilde{\mathscr{P}}^{\infty}$. It follows from (16) that the profile vectors $\theta_{k-L+1}^k(\infty) = 0$, resulting in

$$\mathbb{E}_0\left[S_i^k\right] = -\frac{1}{2}\left[\mathscr{M}\theta_{k-L+1}^k(i)\right]^T\left[\mathscr{W}^T\Sigma^{-1}\mathscr{W}\right]\left[\mathscr{M}\theta_{k-L+1}^k(i)\right], \tag{46}$$

where $\mathbb{E}_0\left[S_i^k\right]$ denotes the mathematical expectation of S_i^k w.r.t. the distribution $\widetilde{\mathscr{P}}_0$.

A.2. Calculation of $\operatorname{cov}\left(S_{i_1}^{k_1}, S_{i_2}^{k_2}\right)$

Under the measure $\widetilde{\mathscr{P}}^{k_0}$, the LLR S_i^k, for $k \geq L$ and $k-L+1 \leq i \leq k$, can be described as a function of the coefficient vector $\phi_{k-L+1}^k(i)$, the random noise vector ξ_{k-L+1}^k and the mathematical expectation $\mathbb{E}_{k_0}\left[S_i^k\right]$ as follows :

$$S_i^k = \left[\phi_{k-L+1}^k(i)\right]^T \xi_{k-L+1}^k + \mathbb{E}_{k_0}\left[S_i^k\right]. \tag{47}$$

Under the pre-change measure $\widetilde{\mathscr{P}}_0$, the LLR S_i^k is described as

$$S_i^k = \left[\phi_{k-L+1}^k(i)\right]^T \xi_{k-L+1}^k + \mathbb{E}_0\left[S_i^k\right]. \tag{48}$$

Let us consider two Gaussian random variables $S_{i_1}^{k_1}$ and $S_{i_2}^{k_2}$, for $k_1,k_2 \geq L$ and $k_1-L+1 \leq i_1 \leq k_1$, $k_2-L+1 \leq i_2 \leq k_2$. It can be seen clearly from (47) – (48) that $\operatorname{cov}\left(S_{i_1}^{k_1}, S_{i_2}^{k_2}\right)$ under the measure $\widetilde{\mathscr{P}}^{k_0}$ is the same as $\operatorname{cov}\left(S_{i_1}^{k_1}, S_{i_2}^{k_2}\right)$ under the measure $\widetilde{\mathscr{P}}_0$. Under the measure $\widetilde{\mathscr{P}}^{k_0}$, we have

$$\begin{cases} S_{i_1}^{k_1} &= \left[\phi_{k_1-L+1}^{k_1}(i_1)\right]^T \xi_{k_1-L+1}^{k_1} + \mathbb{E}_{k_0}\left[S_{i_1}^{k_1}\right] \\ S_{i_2}^{k_2} &= \left[\phi_{k_2-L+1}^{k_2}(i_2)\right]^T \xi_{k_2-L+1}^{k_2} + \mathbb{E}_{k_0}\left[S_{i_2}^{k_2}\right] \end{cases}. \tag{49}$$

It is clear that

$$\mathbb{E}_{k_0}\left[\left[\phi_{k_1-L+1}^{k_1}(i_1)\right]^T \xi_{k_1-L+1}^{k_1}\right] = \mathbb{E}_{k_0}\left[\left[\phi_{k_2-L+1}^{k_2}(i_2)\right]^T \xi_{k_2-L+1}^{k_2}\right] = 0.$$

Then, the covariance is given by

$$\begin{aligned} \operatorname{cov}\left(S_{i_1}^{k_1}, S_{i_2}^{k_2}\right) &= \operatorname{cov}\left(\left[\phi_{k_1-L+1}^{k_1}(i_1)\right]^T \xi_{k_1-L+1}^{k_1}, \left[\phi_{k_2-L+1}^{k_2}(i_2)\right]^T \xi_{k_2-L+1}^{k_2}\right) \\ &= \mathbb{E}_{k_0}\left[\left(\left[\phi_{k_1-L+1}^{k_1}(i_1)\right]^T \xi_{k_1-L+1}^{k_1}\right)\left(\left[\phi_{k_2-L+1}^{k_2}(i_2)\right]^T \xi_{k_2-L+1}^{k_2}\right)\right] \\ &= \mathbb{E}_{k_0}\left[\left(\sum_{t_1=k_1-L+1}^{k_1} \phi_{t_1-k_1+L}^T(i_1)\,\xi_{t_1}\right)\left(\sum_{t_2=k_2-L+1}^{k_2} \phi_{t_2-k_2+L}^T(i_2)\,\xi_{t_2}\right)\right]. \end{aligned} \tag{50}$$

Let $i_{max} = \max(k_1-L+1, k_2-L+1)$ and $k_{min} = \min(k_1,k_2)$, there are two cases :

– If $i_{max} > k_{min}$, then it is clear that $\mathrm{cov}\left(S_{i_1}^{k_1}, S_{i_2}^{k_2}\right) = 0$.
– If $i_{max} \leq k_{min}$, then

$$\begin{aligned}
\mathrm{cov}\left(S_{i_1}^{k_1}, S_{i_2}^{k_2}\right) &= \mathbb{E}_{k_0}\left[\left(\sum_{t_1=i_{max}}^{k_{min}} \phi_{t_1-k_1+L}^T(i_1)\,\xi_{t_1}\right)\left(\sum_{t_2=i_{max}}^{k_{min}} \phi_{t_2-k_2+L}^T(i_2)\,\xi_{t_2}\right)\right] \\
&= \mathbb{E}_{k_0}\left[\sum_{t_1=i_{max}}^{k_{min}}\sum_{t_2=i_{max}}^{k_{min}} \left(\phi_{t_1-k_1+L}^T(i_1)\,\xi_{t_1}\right)\left(\phi_{t_2-k_2+L}^T(i_2)\,\xi_{t_2}\right)\right] \\
&= \mathbb{E}_{k_0}\left[\sum_{t_0=i_{max}}^{k_{min}} \left(\phi_{t_0-k_1+L}^T(i_1)\,\xi_{t_0}\right)\left(\phi_{t_0-k_2+L}^T(i_2)\,\xi_{t_0}\right)\right] \qquad (51)
\end{aligned}$$

since two random noise vectors ξ_{t_1} and ξ_{t_2} are independent for $t_1 \neq t_2$, leading to $\mathbb{E}\left[\xi_{t_i}\xi_{t_j}\right] = 0$, $\forall i \neq j$. Then,

$$\begin{aligned}
\mathrm{cov}\left(S_{i_1}^{k_1}, S_{i_2}^{k_2}\right) &= \sum_{t_0=i_{max}}^{k_{min}} \mathbb{E}_{k_0}\left[\left(\phi_{t_0-k_1+L}^T(i_1)\,\xi_{t_0}\right)\left(\phi_{t_0-k_2+L}^T(i_2)\,\xi_{t_0}\right)\right] \\
&= \sum_{t_0=i_{max}}^{k_{min}} \mathbb{E}_{k_0}\left[\phi_{t_0-k_1+L}^T(i_1)\,\xi_{t_0}\xi_{t_0}^T\phi_{t_0-k_2+L}(i_2)\right] \\
&= \sum_{t_0=i_{max}}^{k_{min}} \left(\phi_{t_0-k_1+L}^T(i_1)\,\mathbb{E}_{k_0}\left[\xi_{t_0}\xi_{t_0}^T\right]\phi_{t_0-k_2+L}(i_2)\right) \\
&= \sum_{t_0=i_{max}}^{k_{min}} \left(\phi_{t_0-k_1+L}^T(i_1)\,R\phi_{t_0-k_2+L}(i_2)\right). \qquad (52)
\end{aligned}$$

The calculation of $\mathbb{E}_{k_0}\left[S_i^k\right]$, $\mathbb{E}_0\left[S_i^k\right]$ and $\mathrm{cov}\left(S_{i_1}^{k_1}, S_{i_2}^{k_2}\right)$ is completed. □.

Appendix B: Proof of Theorem 1

The proof of Theorem 1 is inspired by Guépié et al. (2012b) and Guépié (2013, pages 51-54). In this paper, we generalize the results of Guépié et al. (2012b); Guépié (2013) to a more sophisticated observation model with nuisance parameters (15) – (17). The proof is divided into two parts. In the first part, we investigate the property of the probability of false alarm given in (26). In the second part, we introduce an upper bound for the worst-case probability of missed detection given in (27).

B.1. Proof of part 1

Let us assume the pre-change mode. Let $u_l = \mathbb{P}_0(T_{VTWL} = l)$, we will show that $u_{l+1} \leq u_l$ for all $l \geq L$. For $l = L$, we have

$$\begin{aligned} u_{L+1} &= \mathbb{P}_0(T_{VTWL} = L+1) \\ &= \mathbb{P}_0\left(\left\{\max_{1\leq i\leq L}\left(S_i^L - h_{L-i+1}\right) < 0\right\}\bigcap\left\{\max_{2\leq i\leq L+1}\left(S_i^{L+1} - h_{L-i+2}\right) \geq 0\right\}\right) \\ &\leq \mathbb{P}_0\left(\left\{\max_{2\leq i\leq L+1}\left(S_i^{L+1} - h_{L-i+2}\right) \geq 0\right\}\right). \end{aligned} \tag{53}$$

As it follows from Appendix A, the Gaussian random variables $S_1^L, S_2^L, \cdots, S_L^L$ and $S_2^{L+1}, S_3^{L+1}, \cdots, S_{L+1}^{L+1}$ are generated, respectively, from the random vector $\xi_1^L = (\xi_1^T, \xi_2^T, \cdots, \xi_L^T)^T$ and $\xi_2^{L+1} = (\xi_2^T, \xi_3^T, \cdots, \xi_{L+1}^T)^T$ by (48). It follows from Remark 2 that $\theta_1^L(i) = \theta_2^{L+1}(i+1)$, then $\mathbb{E}_0\left(S_i^L\right) = \mathbb{E}_0\left(S_{i+1}^{L+1}\right)$, for all $1 \leq i \leq L$. The random noises $\xi_1, \xi_2, \cdots, \xi_{L+1}$ are independent identically distributed variables following a zero-mean Gaussian law. Hence, the Gaussian random variables $S_1^L, S_2^L, \cdots, S_L^L$ and $S_2^{L+1}, S_3^{L+1}, \cdots, S_{L+1}^{L+1}$ have the same distributions, leading to

$$\mathbb{P}_0\left(\max_{2\leq i\leq L+1}\left(S_i^{L+1} - h_{L-i+2}\right) \geq 0\right) = \mathbb{P}_0\left(\max_{1\leq i\leq L}\left(S_i^L - h_{L-i+1}\right) \geq 0\right) = u_L. \tag{54}$$

Then, it is clear that $u_{L+1} \leq u_L$. By the same argument, we have for the case $l > L$ that

$$\begin{aligned} u_{l+1} &= \mathbb{P}_0(T_{VTWL} = l+1) \\ &= \mathbb{P}_0\left(\bigcap_{k=L}^{l}\left\{\max_{k-L+1\leq i\leq k}\left(S_i^k - h_{k-i+1}\right) < 0\right\}\bigcap\left\{\max_{l-L+2\leq i\leq l+1}\left(S_i^{l+1} - h_{l-i+2}\right) \geq 0\right\}\right) \\ &\leq \mathbb{P}_0\left(\bigcap_{k=L+1}^{l}\left\{\max_{k-L+1\leq i\leq k}\left(S_i^k - h_{k-i+1}\right) < 0\right\}\bigcap\left\{\max_{l-L+2\leq i\leq l+1}\left(S_i^{l+1} - h_{l-i+2}\right) \geq 0\right\}\right) \\ &\leq \mathbb{P}_0\left(\bigcap_{k=L}^{l-1}\left\{\max_{k-L+1\leq i\leq k}\left(S_i^k - h_{k-i+1}\right) < 0\right\}\bigcap\left\{\max_{l-L+1\leq i\leq l}\left(S_i^{l} - h_{l-i+1}\right) \geq 0\right\}\right) \\ &\leq \mathbb{P}_0(T_{VTWL} = l) = u_l. \end{aligned} \tag{55}$$

Moreover, we obtain from the definition of U_l that

$$U_l - U_{l+1} = \left(\sum_{k=l}^{l+m-1} u_k\right) - \left(\sum_{k=l+1}^{l+m} u_k\right) = u_l - u_{l+m} \geq 0. \tag{56}$$

Hence, $\{U_l\}_{l\geq L}$ is a non-increasing sequence, leading to

$$\overline{\mathbb{P}}_{fa}(T_{VTWL}; m; h_1, h_2, \cdots, h_L) = \sup_{l\geq L}\mathbb{P}_0(l \leq T_{VTWL} \leq l+m-1) = U_L. \tag{57}$$

The proof of part 1 is completed. □.

B.2. *Proof of part 2*

The worst-case probability of missed detection is described as

$$\overline{\mathbb{P}}_{md}(T_{VTWL}; h_1, h_2, \cdots, h_L) = \sup_{k_0 \geq L} \mathbb{P}_{k_0}(T_{VTWL} \geq k_0 + L | T_{VTWL} \geq k_0) = \sup_{k_0 \geq L} \frac{\mathbb{P}_{k_0}(T_{VTWL} \geq k_0 + L)}{\mathbb{P}_{k_0}(T_{VTWL} \geq k_0)}. \quad (58)$$

For the change time $k_0 > L$, the probability of missed detection is given by

$$\mathbb{P}_{k_0}(T_{VTWL} \geq k_0 + L | T_{VTWL} \geq k_0) = \frac{\mathbb{P}_{k_0}\left(\bigcap_{k=L}^{k_0+L-1} \left\{ \max_{k-L+1 \leq i \leq k} \left(S_i^k - h_{k-i+1} \right) < 0 \right\} \right)}{\mathbb{P}_{k_0}\left(\bigcap_{k=L}^{k_0-1} \left\{ \max_{k-L+1 \leq i \leq k} \left(S_i^k - h_{k-i+1} \right) < 0 \right\} \right)}. \quad (59)$$

Let us define three events A_1, A_2 and A_3 as follows:

$$A_1 = \bigcap_{k=L}^{k_0-1} \left\{ \max_{k-L+1 \leq i \leq k} \left(S_i^k - h_{k-i+1} \right) < 0 \right\},$$

$$A_2 = \bigcap_{k=k_0}^{k_0+L-2} \left\{ \max_{k-L+1 \leq i \leq k} \left(S_i^k - h_{k-i+1} \right) < 0 \right\},$$

$$A_3 = \left\{ \max_{k_0 \leq i \leq k_0+L-1} \left(S_i^{k_0+L-1} - h_{k_0+L-i} \right) < 0 \right\}.$$

It follows from (47) that the event A_1 depends on the random vectors $\xi_1^L, \xi_2^{L+1}, \cdots, \xi_{k_0-L}^{k_0-1}$, the event A_2 depends on the random vectors $\xi_{k_0-L+1}^{k_0}, \xi_{k_0-L+2}^{k_0+1}, \cdots, \xi_{k_0-1}^{k_0+L-2}$ and the event A_3 depends on the random vector $\xi_{k_0}^{k_0+L-1}$. Moreover, there is no common elements between the random vectors $\xi_1^L, \xi_2^{L+1}, \cdots, \xi_{k_0-L}^{k_0-1}$ and the random vector $\xi_{k_0}^{k_0+L-1}$. Therefore, the events A_1 and A_3 are independent, leading to

$$\mathbb{P}_{k_0}(T_{VTWL} \geq k_0 + L | T_{VTWL} \geq k_0) = \frac{\mathbb{P}_{k_0}(A_1 \cap A_2 \cap A_3)}{\mathbb{P}_{k_0}(A_1)} \leq \frac{\mathbb{P}_{k_0}(A_1 \cap A_3)}{\mathbb{P}_{k_0}(A_1)} = \mathbb{P}_{k_0}(A_3). \quad (60)$$

For the change time $k_0 = L$, it is clear that

$$\begin{aligned}\mathbb{P}_{k_0}(T_{VTWL} \geq k_0 + L | T_{VTWL} \geq k_0) &= \mathbb{P}_{k_0}(T_{VTWL} \geq k_0 + L) \\ &= \mathbb{P}_{k_0}\left(\bigcap_{k=L}^{k_0+L-1} \left\{ \max_{k-L+1 \leq i \leq k} \left(S_i^k - h_{k-i+1} \right) < 0 \right\} \right) \\ &\leq \mathbb{P}_{k_0}\left(\max_{k_0 \leq i \leq k_0+L-1} \left(S_i^{k_0+L-1} - h_{k_0+L-i} \right) < 0 \right) \\ &\leq \mathbb{P}_{k_0}(A_3). \end{aligned} \quad (61)$$

By replacing the event A_3 with its definition, we get

$$\begin{aligned}\overline{\mathbb{P}}_{md}\left(T_{VTWL};h_1,h_2,\cdots,h_L\right) &\leq \mathbb{P}_{k_0}\left(\max_{k_0\leq i\leq k_0+L-1}\left(S_i^{k_0+L-1}-h_{k_0+L-i}\right)<0\right)\\ &\leq \mathbb{P}_{k_0}\left(\bigcap_{i=k_0}^{k_0+L-1}\left\{S_i^{k_0+L-1}<h_{k_0+L-i}\right\}\right)\\ &\leq \mathbb{P}_{k_0}\left(S_{k_0}^{k_0+L-1}<h_L\right)=\mathbb{P}_1\left(S_1^L<h_L\right)=\widetilde{\mathbb{P}}_{md}\left(T_{VTWL};h_L\right)\end{aligned} \quad (62)$$

where the LLR S_1^L is a Gaussian random variable $S_1^L \sim \mathcal{N}\left(\mu_{S_1^L},\sigma^2_{S_1^L}\right)$ with

$$\mu_{S_1^L} = \mathbb{E}_1\left[S_1^L\right]=\frac{1}{2}\left[\mathscr{M}\theta_1^L(1)\right]^T\left[\mathscr{W}^T\Sigma^{-1}\mathscr{W}\right]\left[\mathscr{M}\theta_1^L(1)\right] \quad (63)$$

$$\sigma^2_{S_1^L} = \operatorname{var}\left(S_1^L\right)=\left[\mathscr{M}\theta_1^L(1)\right]^T\left[\mathscr{W}^T\Sigma^{-1}\mathscr{W}\right]\left[\mathscr{M}\theta_1^L(1)\right]. \quad (64)$$

Finally, the worst-case probability of missed detection is upper bounded by

$$\overline{\mathbb{P}}_{md}\left(T_{VTWL};h_1,h_2,\cdots,h_L\right)\leq\widetilde{\mathbb{P}}_{md}\left(T_{VTWL};h_L\right)\triangleq\Phi\left(\frac{h_L-\mu_{S_1^L}}{\sigma_{S_1^L}}\right). \quad (65)$$

The proof of part 2 is completed. □.

Appendix C: Proof of Lemma 1

Let us suppose that Assumption 1 is satisfied. The goal of this lemma is to show that the covariance matrix $\Sigma_{\mathscr{S}} \in \mathbb{R}^{m\times m}$ of the Gaussian random vector $\mathscr{S} \in \mathbb{R}^m$, which is composed of m Gaussian random variables $S_1^L, S_2^{L+1}, \cdots, S_m^{L+m-1}$, is positive definite. Seeking for simplicity, we eliminate the index "(1)" from $\phi_1^L(1)$ and $\phi_j(1)$ for all $1\leq j\leq L$. As a result, the coefficient vector $\phi_1^L \in \mathbb{R}^{Lp}$ can be rewritten as

$$\phi_1^L=\begin{bmatrix}\phi_1\\ \vdots\\ \phi_L\end{bmatrix}; \quad \phi_j\in\mathbb{R}^p, \quad (66)$$

where p is the dimension of the observation vector y_k in model (11) (corresponding to the dimension of the sensor noises $\{\xi_k\}_{k\geq 1}$). Then, the LLRs $S_1^L, S_2^{L+1}, \cdots, S_m^{L+m-1}$ can be rewritten as

$$\begin{aligned}S_1^L &= \phi_1^T\xi_1+\phi_2^T\xi_2+\cdots+\phi_L^T\xi_L+\mathbb{E}_0\left[S_1^L\right]\\ S_2^{L+1} &= \phi_1^T\xi_2+\phi_2^T\xi_3+\cdots+\phi_L^T\xi_{L+1}+\mathbb{E}_0\left[S_2^{L+1}\right]\\ \vdots\ &\ \vdots\ \ \vdots\\ S_m^{L+m-1} &= \phi_1^T\xi_m+\phi_2^T\xi_{m+1}+\cdots+\phi_L^T\xi_{L+m-1}+\mathbb{E}_0\left[S_m^{L+m-1}\right].\end{aligned}$$

By rewriting the above equation in matrix form, we obtain

$$\underbrace{\begin{bmatrix} S_1^L \\ S_2^{L+1} \\ \vdots \\ S_m^{L+m-1} \end{bmatrix}}_{\mathscr{S}\in\mathbb{R}^m} = \underbrace{\begin{bmatrix} \phi_1^T & \phi_2^T & \cdots & \phi_L^T & 0 & \cdots & 0 \\ 0 & \phi_1^T & \cdots & \phi_{L-1}^T & \phi_L^T & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \cdots & \cdots & \cdots & \phi_L^T \end{bmatrix}}_{\mathscr{Q}\in\mathbb{R}^{m\times(L+m-1)p}} \underbrace{\begin{bmatrix} \xi_1 \\ \xi_2 \\ \vdots \\ \xi_L \\ \vdots \\ \xi_{L+m-1} \end{bmatrix}}_{\xi_1^{L+m-1}\in\mathbb{R}^{(L+m-1)p}} + \underbrace{\begin{bmatrix} \mathbb{E}_0\left[S_1^L\right] \\ \mathbb{E}_0\left[S_2^{L+1}\right] \\ \vdots \\ \mathbb{E}_0\left[S_m^{L+m-1}\right] \end{bmatrix}}_{\mu_{\mathscr{S}}\in\mathbb{R}^m}. \tag{67}$$

It is worth noting that the random noise vector $\xi_1^{L+m-1} \sim \mathscr{N}\left(0,\widetilde{\Sigma}\right)$, where $\widetilde{\Sigma}\in\mathbb{R}^{(L+m-1)p\times(L+m-1)p}$ is a positive definite matrix since the sensor noises $\xi_1,\xi_2,\cdots,\xi_{L+m-1}$ are independent identically distributed zero-mean normal variables. Then, the covariance matrix $\Sigma_{\mathscr{S}}$ of the normal random vector $\mathscr{S}$ is calculated by

$$\Sigma_{\mathscr{S}} \;\; = \;\; \mathscr{Q}\widetilde{\Sigma}\mathscr{Q}^T \tag{68}$$

In the following, we will show that the matrix $\Sigma_{\mathscr{S}}$ defined in (68) is positive definite. By Assumption 1, there exists at least one vector $\phi_j^T \neq 0$, for $1\leq j\leq L$.

Firstly, if the coefficient vector $\phi_1^T = \left[\phi_1^1,\cdots,\phi_1^p\right] \neq 0$, then there exists at least one element $\phi_1^i \neq 0$, for $1\leq i\leq p$. Let $M_1\in\mathbb{R}^{m\times m}$ be a square matrix formulated by m columns containing the element $\phi_1^i \neq 0$ from the matrix $\mathscr{Q}$. Then, the matrix M_1 can be described as

$$M_1 = \begin{bmatrix} \phi_1^i & - & \cdots & - \\ 0 & \phi_1^i & \cdots & - \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \phi_1^i \end{bmatrix} \tag{69}$$

where the notation "–" stands for any real number. Matrix M_1 is an upper triangular one with non-zero elements in the diagonal (i.e., $\phi_1^i \neq 0$), then $\text{rank}\,(M_1) = m$. Since the columns of matrix M_1 are contained in matrix $\mathscr{Q}$ and matrix $\mathscr{Q}$ has m rows, we have $\text{rank}\,(\mathscr{Q}) = m$.

Secondly, if the coefficient vector $\phi_1^T = 0$ and the coefficient vector $\phi_2^T \neq 0$. Then, there exists at least one element $\phi_2^i \neq 0$, for $1\leq i\leq p$. Let $M_2\in\mathbb{R}^{m\times m}$ be a square matrix formulated by m columns containing the element $\phi_2^i \neq 0$ from the matrix $\mathscr{Q}$. By the same argument as above, we obtain $\text{rank}\,(\mathscr{Q}) = m$.

By the same procedure, it can be concluded that matrix $\mathscr{Q}$ is full row rank (i.e., $\text{rank}\,(\mathscr{Q}) = m$) if Assumption 1 is satisfied. As it follows from (Koch, 1999, page 47) that, if matrix $\mathscr{Q}$ is full row rank, the covariance matrix $\Sigma_{\mathscr{S}}$ in (68) is positive definite. □.

Appendix D: Proof of Theorem 2

The proof of Theorem 2 consists of two parts. The optimization problem is formulated and solved in the first part. It is shown in the second part that the optimized VTWL CUSUM test is equivalent to the FMA test.

D.1. Proof of part 1

Since we wish to minimize the upper bound $\widetilde{\mathbb{P}}_{md}(T_{VTWL};h_L)$ on the worst-case probability of missed detection $\overline{\mathbb{P}}_{md}(T_{VTWL};h_1,h_2,\cdots,h_L)$ subject to an acceptable level $\alpha \in (0,1)$ on the worst-case probability of false alarm, the optimization problem can be defined as

$$\begin{cases} \inf\limits_{h_1,h_2,\ldots,h_L} \left\{\widetilde{\mathbb{P}}_{md}(T_{VTWL};h_L)\right\} \\ \text{subject to } \overline{\mathbb{P}}_{fa}(T_{VTWL};m;h_1,h_2,\cdots,h_L) \leq \alpha, \end{cases} \tag{70}$$

where the worst-case probability of false alarm $\overline{\mathbb{P}}_{fa}(T_{VTWL};m;h_1,h_2,\cdots,h_L)$ is given by

$$\begin{aligned} \overline{\mathbb{P}}_{fa}(T_{VTWL};m;h_1,h_2,\cdots,h_L) &= \mathbb{P}_0(L \leq T_{VTWL} \leq L+m-1) \\ &= 1-\mathbb{P}_0(T_{VTWL} \geq L+m) \\ &= 1-\mathbb{P}_0\left(\bigcap_{k=L}^{L+m-1}\left\{\max_{k-L+1\leq i\leq k}\left(S_i^k - h_{k-i+1}\right) < 0\right\}\right) \\ &= 1-\mathbb{P}_0\left(\bigcap_{k=L}^{L+m-1}\bigcap_{i=k-L+1}^{k}\left\{S_i^k < h_{k-i+1}\right\}\right). \end{aligned} \tag{71}$$

Seeking for simplicity, let us define a function $F_0(h_1,h_2,\cdots,h_{L-1},h_L)$ as follows:

$$F_0(h_1,h_2,\cdots,h_{L-1},h_L) = \mathbb{P}_0\left(\bigcap_{k=L}^{L+m-1}\bigcap_{i=k-L+1}^{k}\left\{S_i^k < h_{k-i+1}\right\}\right). \tag{72}$$

The optimization problem (70) becomes

$$\begin{cases} \inf\limits_{h_1,h_2,\ldots,h_L} \left\{\widetilde{\mathbb{P}}_{md}(T_{VTWL};h_L)\right\} \\ \text{subject to } F_0(h_1,h_2,\cdots,h_{L-1},h_L) \geq 1-\alpha, \end{cases} \tag{73}$$

where the objective function $\widetilde{\mathbb{P}}_{md}(T_{VTWL};h_L) = \Phi\left(\frac{h_L-\mu_{S_1^L}}{\sigma_{S_1^L}}\right)$ is monotonically non-decreasing w.r.t. the threshold h_L.

Before solving the optimization problem (73), let us prove that the function $F_0(h_1,h_2,\cdots,h_{L-1},h_L)$ is monotonically non-decreasing w.r.t. each threshold $h_1,h_2,\cdots,h_{L-1},h_L$. Let $\left\{\delta h_j\right\}_{1\leq j\leq L}$ be positive real numbers, then

$$\begin{aligned}
F_0\left(h_1,\cdots,h_j+\delta h_j,\cdots,h_L\right) &= \mathbb{P}_0\left(\bigcap_{k=L}^{L+m-1}\left[\bigcap_{\substack{i=k-L+1\\ i\neq k-j+1}}^{k}\left\{S_i^k<h_{k-i+1}\right\} \text{ and } \left\{S_{k-j+1}^k<h_j+\delta h_j\right\}\right]\right)\\
&= \mathbb{P}_0\left(\bigcap_{k=L}^{L+m-1}\left[\left\{\bigcap_{\substack{i=k-L+1\\ i\neq k-j+1}}^{k}\left\{S_i^k<h_{k-i+1}\right\} \text{ and } \left\{S_{k-j+1}^k<h_j\right\}\right\}\bigcup\right.\right.\\
&\qquad \left.\left.\left\{\bigcap_{\substack{i=k-L+1\\ i\neq k-j+1}}^{k}\left\{S_i^k<h_{k-i+1}\right\} \text{ and } \left\{h_j\leq S_{k-j+1}^k<h_j+\delta h_j\right\}\right\}\right]\right)\\
&\geq \mathbb{P}_0\left(\bigcap_{k=L}^{L+m-1}\left[\bigcap_{\substack{i=k-L+1\\ i\neq k-j+1}}^{k}\left\{S_i^k<h_{k-i+1}\right\} \text{ and } \left\{S_{k-j+1}^k<h_j\right\}\right]\right)\\
&\geq \mathbb{P}_0\left(\bigcap_{k=L}^{L+m-1}\left[\bigcap_{i=k-L+1}^{k}\left\{S_i^k<h_{k-i+1}\right\}\right]\right)\\
&\geq F_0\left(h_1,\cdots,h_j,\cdots,h_L\right).
\end{aligned} \tag{74}$$

Hence, the function $F_0\left(h_1,h_2,\cdots,h_{L-1},h_L\right)$ is monotonically non-decreasing w.r.t. each threshold $h_1,h_2,\cdots,h_{L-1},h_L$. By using this property, we prove in the following that the thresholds $h_1^*,h_2^*,\cdots,h_{L-1}^*\to+\infty$ and the threshold h_L^* satisfying

$$F_0\left(+\infty,+\infty,\cdots,+\infty,h_L^*\right)=\mathbb{P}_0\left(\bigcap_{k=L}^{L+m-1}\left\{S_{k-L+1}^k<h_L^*\right\}\right)=1-\alpha \tag{75}$$

are the solution to the optimization problem (73). The proof consists of two following steps:

1. As it follows from Lemma 1, the covariance matrix $\Sigma_{\mathscr{S}}$ of the Gaussian random variables $S_1^L,S_2^{L+1},\cdots,S_m^{L+m-1}$ is positive definite. The function $F_0\left(+\infty,+\infty,\cdots,+\infty,h_L^*\right)$ is monotonically non-decreasing w.r.t. the threshold h_L^*. Its codomain is $[0,1]$. Hence, equation (75) has a unique solution h_L^* for a given value $\alpha\in(0,1)$.
2. Let us suppose that a set of thresholds $h_1,h_2,\cdots,h_{L-1},h_L$, satisfying the constraint

$$F_0\left(h_1,h_2,\cdots,h_{L-1},h_L\right)\geq 1-\alpha, \tag{76}$$

defines any alternative solution of optimization problem (73). The goal is to show that $\widetilde{\mathbb{P}}_{md}\left(T_{VTWL};h_L\right)\geq\widetilde{\mathbb{P}}_{md}\left(T_{VTWL};h_L^*\right)$.

It follows from the monotonically non-decreasing property of the function $F_0\left(\cdot\right)$ that

$$1-\alpha=F_0\left(+\infty,+\infty,\cdots,+\infty,h_L^*\right)\geq F_0\left(h_1,h_2,\cdots,h_{L-1},h_L^*\right). \tag{77}$$

Putting together (76) and (77), we get the following

$$F_0\left(h_1,h_2,\cdots,h_{L-1},h_L\right)\geq F_0\left(h_1,h_2,\cdots,h_{L-1},h_L^*\right). \tag{78}$$

Hence, $h_L \geq h_L^*$. It follows from (27) that the objective function $h_L \mapsto \widetilde{\mathbb{P}}_{md}(T_{VTWL}; h_L)$ is monotonically non-decreasing. Therefore, $\widetilde{\mathbb{P}}_{md}(T_{VTWL}; h_L) \geq \widetilde{\mathbb{P}}_{md}(T_{VTWL}; h_L^*)$.

The proof of part 1 is completed. □.

D.2. *Proof of part 2*

The VTWL CUSUM test with optimal thresholds $h_1^*, h_2^*, \cdots, h_L^*$ is given by

$$\begin{aligned} T_{VTWL}^* &= \inf\left\{k \geq L : \max_{k-L+1 \leq i \leq k} \left(S_i^k - h_{k-i+1}^*\right) \geq 0\right\} \\ &= \inf\left\{k \geq L : S_{k-L+1}^k \geq h_L^*\right\} \end{aligned} \tag{79}$$

since the optimal thresholds $h_1^*, h_2^*, \cdots, h_{L-1}^* \to +\infty$. In addition, the LLR S_{k-L+1}^k can be rewritten as

$$S_{k-L+1}^k = \left[\mathscr{M}\theta_1^L(1)\right]^T \left[\mathscr{W}^T \Sigma^{-1} \mathscr{W}\right] \left[r_{k-L+1}^k - \frac{1}{2}\mathscr{M}\theta_1^L(1)\right]. \tag{80}$$

Then, optimized VTWL CUSUM test

$$\begin{aligned} T_{VTWL}^* &= \inf\left\{k \geq L : \left[\mathscr{M}\theta_1^L(1)\right]^T \left[\mathscr{W}^T \Sigma^{-1} \mathscr{W}\right] \left[r_{k-L+1}^k - \frac{1}{2}\mathscr{M}\theta_1^L(1)\right] \geq h_L^*\right\} \\ &= \inf\left\{k \geq L : \left[\mathscr{M}\theta_1^L(1)\right]^T \left[\mathscr{W}^T \Sigma^{-1} \mathscr{W}\right] r_{k-L+1}^k \geq \widetilde{h}_L\right\}, \end{aligned} \tag{81}$$

where the threshold

$$\widetilde{h}_L = h_L^* + \mu_{S_1^L}. \tag{82}$$

It is clear that the VTWL CUSUM test with optimal thresholds $h_1^*, h_2^*, \cdots, h_L^*$ is equivalent to the following FMA test:

$$T_{FMA}\left(\widetilde{h}_L\right) = \inf\left\{k \geq L : \left[\mathscr{M}\theta_1^L(1)\right]^T \left[\mathscr{W}^T \Sigma^{-1} \mathscr{W}\right] r_{k-L+1}^k \geq \widetilde{h}_L\right\} \tag{83}$$

and the upper bound on the worst-case probability of missed detection of the FMA test can be expressed as a function of the threshold $\widetilde{h}_L$ by

$$\overline{\mathbb{P}}_{md}\left(T_{FMA}; \widetilde{h}_L\right) = \overline{\mathbb{P}}_{md}\left(T_{VTWL}^*; h_L^*\right) \leq \widetilde{\mathbb{P}}_{md}\left(T_{VTWL}^*; h_L^*\right) = \widetilde{\mathbb{P}}_{md}\left(T_{FMA}; \widetilde{h}_L\right) = \Phi\left(\frac{\widetilde{h}_L - 2\mu_{S_1^L}}{\sigma_{S_1^L}}\right). \tag{84}$$

The proof of part 2 is completed. □.

Appendix E: Proof of Proposition 1

The proof of Proposition 1 is divided into four parts. In each part, we formulate the threshold vector, the mean vector, the covariance matrix for calculating the probability of false alarm and the probability of missed detection for the VTWL CUSUM and the FMA tests.

E.1. Numerical calculation of $\overline{\mathbb{P}}_{fa}(T_{VTWL};m;h_1,h_2,\cdots,h_L)$

The worst-case probability of false alarm of the VTWL CUSUM test is given by

$$\begin{aligned}\overline{\mathbb{P}}_{fa}(T_{VTWL};m;h_1,h_2,\cdots,h_L) &= 1-\mathbb{P}_0\left(\bigcap_{k=L}^{L+m-1}\left\{\max_{k-L+1\le i\le k}\left(S_i^k-h_{k-i+1}\right)<0\right\}\right)\\ &= 1-\mathbb{P}_0\left(\underbrace{\bigcap_{k=L}^{L+m-1}\bigcap_{i=k-L+1}^{k}\left\{S_i^k<h_{k-i+1}\right\}}_{E_1}\right),\end{aligned} \tag{85}$$

where the event E_1 can be rewritten as follows:

$$E_1=\left(\begin{array}{ccccccccc}\left\{S_1^L<h_L\right\} & \bigcap & \left\{S_2^L<h_{L-1}\right\} & \bigcap & \cdots & \bigcap & \left\{S_L^L<h_1\right\} & \bigcap\\ \left\{S_2^{L+1}<h_L\right\} & \bigcap & \left\{S_3^{L+1}<h_{L-1}\right\} & \bigcap & \cdots & \bigcap & \left\{S_{L+1}^{L+1}<h_1\right\} & \bigcap\\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots\\ \left\{S_m^{L+m-1}<h_L\right\} & \bigcap & \left\{S_{m+1}^{L+m-1}<h_{L-1}\right\} & \bigcap & \cdots & \bigcap & \left\{S_{L+m-1}^{L+m-1}<h_1\right\} & \end{array}\right)$$

is composed of m rows and L columns. By organizing the event E_1 in column-by-column manner, the multivariate Gaussian random variable $\mathscr{S}_1\in\mathbb{R}^{mL}$ with the mean vector $\mu_{\mathscr{S}_1}\in\mathbb{R}^{mL}$ and the covariance matrix $\Sigma_{\mathscr{S}_1}\in\mathbb{R}^{mL\times mL}$ and the corresponding threshold vector $h_{\mathscr{S}_1}\in\mathbb{R}^{mL}$ are described as follows:

$$\begin{aligned}\mathscr{S}_1 &= \begin{bmatrix}S_1^L\\ S_2^{L+1}\\ \vdots\\ S_{L+m-1}^{L+m-1}\end{bmatrix};\quad h_{\mathscr{S}_1}=\begin{bmatrix}h_L\\ h_L\\ \vdots\\ h_1\end{bmatrix};\quad \mu_{\mathscr{S}_1}=\begin{bmatrix}\mathbb{E}_0\left[S_1^L\right]\\ \mathbb{E}_0\left[S_2^{L+1}\right]\\ \vdots\\ \mathbb{E}_0\left[S_{L+m-1}^{L+m-1}\right]\end{bmatrix}\\ \Sigma_{\mathscr{S}_1} &= \begin{bmatrix}\mathrm{cov}\left(S_1^L,S_1^L\right) & \mathrm{cov}\left(S_1^L,S_2^{L+1}\right) & \cdots & \mathrm{cov}\left(S_1^L,S_{L+m-1}^{L+m-1}\right)\\ \mathrm{cov}\left(S_2^{L+1},S_1^L\right) & \mathrm{cov}\left(S_2^{L+1},S_2^{L+1}\right) & \cdots & \mathrm{cov}\left(S_2^{L+1},S_{L+m-1}^{L+m-1}\right)\\ \vdots & \vdots & \ddots & \vdots\\ \mathrm{cov}\left(S_{L+m-1}^{L+m-1},S_1^L\right) & \mathrm{cov}\left(S_{L+m-1}^{L+m-1},S_2^{L+1}\right) & \cdots & \mathrm{cov}\left(S_{L+m-1}^{L+m-1},S_{L+m-1}^{L+m-1}\right)\end{bmatrix},\end{aligned}$$

where $\mathbb{E}_0\left[S_i^k\right]$ and $\mathrm{cov}\left(S_{i_1}^{k_1},S_{i_2}^{k_2}\right)$ have been calculated in Appendix A. Then, the worst-case probability of false alarm for the VTWL CUSUM test is calculated as

$$\overline{\mathbb{P}}_{fa}(T_{VTWL};m;h_1,h_2,\cdots,h_L)=\mathbb{P}\left(\bigcap_{j=1}^{mL}\left\{\mathscr{S}_1(j)<h_{\mathscr{S}_1}(j)\right\}\right) \tag{86}$$

The proof of part 1 is completed. □.

E.2. Numerical calculation of $\overline{\mathbb{P}}_{fa}\left(T_{FMA};m;\widetilde{h}_L\right)$

The worst-case probability of false alarm of the FMA test

$$\begin{aligned}\overline{\mathbb{P}}_{fa}\left(T_{FMA};m;\widetilde{h}_L\right) &= \overline{\mathbb{P}}_{fa}\left(T_{VTWL};m;+\infty,+\infty,\cdots,+\infty,\widetilde{h}_L-\mu_{S_1^L}\right) \\ &= 1-\mathbb{P}_0\left(\underbrace{\bigcap_{k=L}^{L+m-1}\left\{S_{k-L+1}^k<\widetilde{h}_L-\mu_{S_1^L}\right\}}_{E_2}\right), \end{aligned} \tag{87}$$

where the event E_2 is defined by m Gaussian random variables $S_1^L,S_2^{L+1},\cdots,S_m^{L+m-1}$. The Gaussian vector $\mathscr{S}_2\in\mathbb{R}^m$ has the mean vector $\mu_{\mathscr{S}_2}\in\mathbb{R}^m$ and the covariance matrix $\Sigma_{\mathscr{S}_2}\in\mathbb{R}^{m\times m}$. The vector $h_{\mathscr{S}_2}\in\mathbb{R}^m$ denotes corresponding thresholds. Hence, we get

$$\begin{aligned}\mathscr{S}_2 &= \begin{bmatrix} S_1^L \\ S_2^{L+1} \\ \vdots \\ S_m^{L+m-1}\end{bmatrix}; \quad h_{\mathscr{S}_2}=\begin{bmatrix}\widetilde{h}_L-\mu_{S_1^L} \\ \widetilde{h}_L-\mu_{S_1^L} \\ \vdots \\ \widetilde{h}_L-\mu_{S_1^L}\end{bmatrix}; \quad \mu_{\mathscr{S}_2}=\begin{bmatrix}\mathbb{E}_0\left[S_1^L\right] \\ \mathbb{E}_0\left[S_2^{L+1}\right] \\ \vdots \\ \mathbb{E}_0\left[S_m^{L+m-1}\right]\end{bmatrix} \\ \Sigma_{\mathscr{S}_2} &= \begin{bmatrix} \mathrm{cov}\left(S_1^L,S_1^L\right) & \mathrm{cov}\left(S_1^L,S_2^{L+1}\right) & \cdots & \mathrm{cov}\left(S_1^L,S_m^{L+m-1}\right) \\ \mathrm{cov}\left(S_2^{L+1},S_1^L\right) & \mathrm{cov}\left(S_2^{L+1},S_2^{L+1}\right) & \cdots & \mathrm{cov}\left(S_2^{L+1},S_m^{L+m-1}\right) \\ \vdots & \vdots & \ddots & \vdots \\ \mathrm{cov}\left(S_m^{L+m-1},S_1^L\right) & \mathrm{cov}\left(S_m^{L+m-1},S_2^{L+1}\right) & \cdots & \mathrm{cov}\left(S_m^{L+m-1},S_m^{L+m-1}\right)\end{bmatrix}.\end{aligned}$$

Then, the worst-case probability of false alarm of the FMA test is calculated numerically as

$$\overline{\mathbb{P}}_{fa}\left(T_{FMA};m;\widetilde{h}_L\right)=\mathbb{P}\left(\bigcap_{j=1}^{m}\left\{\mathscr{S}_2(j)<h_{\mathscr{S}_2}(j)\right\}\right) \tag{88}$$

The proof of part 2 is completed. □.

E.3. Numerical calculation of $\mathbb{P}_{k_0}\left(T_{VTWL}\geq k_0+L|T_{VTWL}\geq k_0\right)$

Let us define the following function $F_{k_0}(a;b;h_1,h_2,\cdots,h_L)$ with $b\geq a\geq L$, $a,b\in\mathbb{N}$:

$$\begin{aligned}F_{k_0}(a;b;h_1,h_2,\cdots,h_L) &= \mathbb{P}_{k_0}\left(\bigcap_{k=a}^{b}\left\{\max_{k-L+1\leq i\leq k}\left(S_i^k-h_{k-i+1}\right)<0\right\}\right) \\ &= \mathbb{P}_{k_0}\left(\underbrace{\bigcap_{k=a}^{b}\bigcap_{i=k-L+1}^{k}\left\{S_i^k<h_{k-i+1}\right\}}_{E_3}\right), \end{aligned} \tag{89}$$

where the event E_3 can be rewritten as follows

$$E_3 = \begin{pmatrix} \{S^a_{a-L+1} < h_L\} & \bigcap & \{S^a_{a-L+2} < h_{L-1}\} & \bigcap & \cdots & \bigcap & \{S^a_a < h_1\} & \bigcap \\ \{S^{a+1}_{a-L+2} < h_L\} & \bigcap & \{S^{a+1}_{a-L+3} < h_{L-1}\} & \bigcap & \cdots & \bigcap & \{S^{a+1}_{a+1} < h_1\} & \bigcap \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \{S^b_{b-L+1} < h_L\} & \bigcap & \{S^b_{b-L+2} < h_{L-1}\} & \bigcap & \cdots & \bigcap & \{S^b_b < h_1\} & \end{pmatrix}$$

, is comprised of $b-a+1$ rows and L columns. Then, the multivariate Gaussian random variable $\mathscr{S}_3 \in \mathbb{R}^{(b-a+1)L}$ with the mean vector $\mu_{\mathscr{S}_3} \in \mathbb{R}^{(b-a+1)L}$ and the covariance matrix $\Sigma_{\mathscr{S}_3} \in \mathbb{R}^{(b-a+1)L\times(b-a+1)L}$ and the corresponding threshold vector $h_{\mathscr{S}_3}$ can be described as

$$\mathscr{S}_3 = \begin{bmatrix} S^a_{a-L+1} \\ S^{a+1}_{a-L+2} \\ \vdots \\ S^b_b \end{bmatrix}; \quad h_{\mathscr{S}_3} = \begin{bmatrix} h_L \\ h_L \\ \vdots \\ h_1 \end{bmatrix}; \quad \mu_{\mathscr{S}_3} = \begin{bmatrix} \mathbb{E}_{k_0}\left[S^a_{a-L+1}\right] \\ \mathbb{E}_{k_0}\left[S^{a+1}_{a-L+2}\right] \\ \vdots \\ \mathbb{E}_{k_0}\left[S^b_b\right] \end{bmatrix}$$

$$\Sigma_{\mathscr{S}_3} = \begin{bmatrix} \operatorname{cov}\left(S^a_{a-L+1}, S^a_{a-L+1}\right) & \operatorname{cov}\left(S^a_{a-L+1}, S^{a+1}_{a-L+2}\right) & \cdots & \operatorname{cov}\left(S^a_{a-L+1}, S^b_b\right) \\ \operatorname{cov}\left(S^{a+1}_{a-L+2}, S^a_{a-L+1}\right) & \operatorname{cov}\left(S^{a+1}_{a-L+2}, S^{a+1}_{a-L+2}\right) & \cdots & \operatorname{cov}\left(S^{a+2}_{a-L+2}, S^b_b\right) \\ \vdots & \vdots & \ddots & \vdots \\ \operatorname{cov}\left(S^b_b, S^a_{a-L+1}\right) & \operatorname{cov}\left(S^b_b, S^{a+1}_{a-L+2}\right) & \cdots & \operatorname{cov}\left(S^b_b, S^b_b\right) \end{bmatrix}$$

Then, the function $F_{k_0}(a;b;h_1,h_2,\cdots,h_L)$ can be calculated numerically as

$$F_{k_0}(a;b;h_1,h_2,\cdots,h_L) = \mathbb{P}_{k_0}\left(\bigcap_{j=1}^{(b-a+1)L}\left\{\mathscr{S}_3(j) < h_{\mathscr{S}_3}(j)\right\}\right) \tag{90}$$

Finally, the worst-case probability of missed detection for the VTWL CUSUM test can be calculated as

$$\mathbb{P}_{k_0}(T_{VTWL} \geq k_0+L|T_{VTWL} \geq k_0) = \frac{F_{k_0}(L;k_0+L-1;h_1,h_2,\cdots,h_L)}{F_{k_0}(L;k_0-1;h_1,h_2,\cdots,h_L)}, \tag{91}$$

where $F_{k_0}(L;k_0-1;h_1,h_2,\cdots,h_L) \triangleq 1$ for $k_0 = L$. The proof of part 3 is completed. □.

E.4. Numerical calculation of $\mathbb{P}_{k_0}(T_{FMA} \geq k_0+L|T_{FMA} \geq k_0)$

Let us define the following function $\widetilde{F}_{k_0}\left(a;b;\widetilde{h}_L\right)$

$$\widetilde{F}_{k_0}\left(a;b;\widetilde{h}_L\right) = \mathbb{P}_{k_0}\left(\bigcap_{k=a}^{b}\left\{S^k_{k-L+1} < \widetilde{h}_L\right\}\right). \tag{92}$$

The multivariate Gaussian random variable $\mathscr{S}_4 \in \mathbb{R}^{(b-a+1)}$ with its mean vector $\mu_{\mathscr{S}_4} \in \mathbb{R}^{(b-a+1)}$, covariance matrix $\Sigma_{\mathscr{S}_4} \in \mathbb{R}^{(b-a+1)\times(b-a+1)}$ and the threshold vector $h_{\mathscr{S}_4} \in \mathbb{R}^{(b-a+1)}$ is defined as follows

$$\mathscr{S}_4 = \begin{bmatrix} S^a_{a-L+1} \\ S^{a+1}_{a-L+2} \\ \vdots \\ S^b_{b-L+1} \end{bmatrix}; \quad \mu_{\mathscr{S}_4} = \begin{bmatrix} \mathbb{E}_{k_0}\left[S^a_{a-L+1}\right] \\ \mathbb{E}_{k_0}\left[S^{a+1}_{a-L+2}\right] \\ \vdots \\ \mathbb{E}_{k_0}\left[S^b_{b-L+1}\right] \end{bmatrix}; \quad h_{\mathscr{S}_4} = \begin{bmatrix} \widetilde{h}_L \\ \widetilde{h}_L \\ \vdots \\ \widetilde{h}_L \end{bmatrix}$$

$$\Sigma_{\mathscr{S}_4} = \begin{bmatrix} \mathrm{cov}\left(S^a_{a-L+1}, S^a_{a-L+1}\right) & \mathrm{cov}\left(S^a_{a-L+1}, S^{a+1}_{a-L+2}\right) & \cdots & \mathrm{cov}\left(S^a_{a-L+1}, S^b_{b-L+1}\right) \\ \mathrm{cov}\left(S^{a+1}_{a-L+2}, S^a_{a-L+1}\right) & \mathrm{cov}\left(S^{a+1}_{a-L+2}, S^{a+1}_{a-L+2}\right) & \cdots & \mathrm{cov}\left(S^{a+1}_{a-L+2}, S^b_{b-L+1}\right) \\ \vdots & \vdots & \ddots & \vdots \\ \mathrm{cov}\left(S^b_{b-L+1}, S^a_{a-L+1}\right) & \mathrm{cov}\left(S^b_{b-L+1}, S^{a+1}_{a-L+2}\right) & \cdots & \mathrm{cov}\left(S^b_{b-L+1}, S^b_{b-L+1}\right) \end{bmatrix}.$$

The function $\widetilde{F}_{k_0}\left(a;b;\widetilde{h}_L\right)$ is calculated numerically as

$$\widetilde{F}_{k_0}\left(a;b;\widetilde{h}_L\right) = \mathbb{P}_{k_0}\left(\bigcap_{j=1}^{(b-a+1)} \left\{\mathscr{S}_4(j) < h_{\mathscr{S}_4}(j)\right\}\right). \tag{93}$$

Finally, the worst-case probability of missed detection for the FMA test is calculated numerically as

$$\begin{aligned} \mathbb{P}_{k_0}\left(T_{FMA} \geq k_0 + L | T_{FMA} \geq k_0\right) &= \frac{\mathbb{P}_{k_0}\left(\bigcap_{k=L}^{k_0+L-1} \left\{S^k_{k-L+1} < \widetilde{h}_L - \mu_{S^L_1}\right\}\right)}{\mathbb{P}_{k_0}\left(\bigcap_{k=L}^{k_0-1} \left\{S^k_{k-L+1} < \widetilde{h}_L - \mu_{S^L_1}\right\}\right)} \\ &= \frac{\widetilde{F}_{k_0}\left(L; k_0+L-1; \widetilde{h}_L - \mu_{S^L_1}\right)}{\widetilde{F}_{k_0}\left(L; k_0-1; \widetilde{h}_L - \mu_{S^L_1}\right)}, \end{aligned} \tag{94}$$

where $\widetilde{F}_{k_0}\left(L; k_0-1; \widetilde{h}_L - \mu_{S^L_1}\right) \triangleq 1$ for $k_0 = L$. The proof of part 4 is completed. □.

Remark 7. *Let us discuss now the positive definiteness of the covariance matrices $\Sigma_{\mathscr{S}_1}$, $\Sigma_{\mathscr{S}_2}$, $\Sigma_{\mathscr{S}_3}$, and $\Sigma_{\mathscr{S}_4}$. First of all, $\mathscr{S}_2 = \mathscr{S}$, where the last vector is defined in Lemma 1. As it follows from Lemma 1, the covariance matrices $\Sigma_{\mathscr{S}_2} = \Sigma_{\mathscr{S}}$, which correspond to the FMA test, are positive definite if Assumption 1 is satisfied. Second, the covariance matrix $\Sigma_{\mathscr{S}_4}$ is also positive definite if Assumption 1 is satisfied. The proof of this fact is completely analogous to that of Lemma 1 in Appendix C. Finally, the covariance matrices $\Sigma_{\mathscr{S}_1}$ and $\Sigma_{\mathscr{S}_3}$, which correspond to the VTWL CUSUM test, are positive definite in some scenarios. Nevertheless, there are also scenarios where the determinants of $\Sigma_{\mathscr{S}_1}$ and $\Sigma_{\mathscr{S}_3}$ are close to zero, especially with large values of L and m. Therefore, it is necessary to verify their positive definiteness before executing the numerical computation. The following heuristic solution is proposed in such cases : to use the matrix $\Sigma_{\mathscr{S}_1} + \delta\mathscr{I}$ (resp. $\Sigma_{\mathscr{S}_3} + \delta\mathscr{I}$) instead of covariance matrix $\Sigma_{\mathscr{S}_1}$ (resp. $\Sigma_{\mathscr{S}_3}$), where $\mathscr{I}$ is the identity matrix and $\delta > 0$ is a small quantity.*

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References

Amin, S., Litrico, X., Sastry, S., and Bayen, A. (2012a). Cyber security of water SCADA systems:(i) analysis and experimentation of stealthy deception attacks. *IEEE Transactions on Control Systems Technology*, 20.

Amin, S., Litrico, X., Sastry, S., and Bayen, A. (2012b). Cyber security of water SCADA systems:(ii) attack detection using an enhanced hydrodynamic model. *IEEE Transactions on Control Systems Technology*, 20.

Bakhache, B. and Nikiforov, I. (2000). Reliable detection of faults in measurement systems. *International Journal of adaptive control and signal processing*, 14(7):683–700.

Basseville, M. and Nikiforov, I. V. (1993). *Detection of Abrupt Changes - Theory and Application*. Information and System Sciences Series. Prentice-Hall, Inc., Englewood Cliffs, NJ, USA. Online.

Brunner, M., Hofinger, H., Krauss, C., Roblee, C., Schoo, P., and Todt, S. (2010). Infiltrating critical infrastructures with next-generation attacks. *Fraunhofer Institute for Secure Information Technology (SIT), Munich.*

Cárdenas, A. A., Amin, S., Lin, Z.-S., Huang, Y.-L., Huang, C.-Y., and Sastry, S. (2011). Attacks against process control systems: risk assessment, detection, and response. In *Proceedings of the 6th ACM Symposium on Information, Computer and Communications Security*, pages 355–366. ACM.

Chen, B. and Willett, P. (2000). Detection of hidden Markov model transient signals. *IEEE Transactions on Aerospace and Electronic Systems*, 36(4):1253–1268.

Fillatre, L. and Nikiforov, I. (2007). Non-bayesian detection and detectability of anomalies from a few noisy tomographic projections. *Signal Processing, IEEE Transactions on*, 55(2):401–413.

Fouladirad, M. and Nikiforov, I. (2005). Optimal statistical fault detection with nuisance parameters. *Automatica*, 41(7):1157–1171.

Genz, A. and Bretz, F. (2002). Comparison of methods for the computation of multivariate t probabilities. *Journal of Computational and Graphical Statistics*, 11(4):950–971.

Guépié, B. K. (2013). *Détection séquentielle de signaux transitoires : application à la surveillance d'un réseau d'eau potable*. PhD thesis, Université de Technologie de Troyes.

Guépié, B. K., Fillatre, L., and Nikiforov, I. (2012a). Detecting an abrupt change of finite duration. In *Signals, Systems and Computers (ASILOMAR), 2012 Conference Record of the Forty Sixth Asilomar Conference on*, pages 1930–1934. IEEE.

Guépié, B. K., Fillatre, L., and Nikiforov, I. (2012b). Sequential detection of transient changes. *Sequential Analysis*, 31(4):528–547.

Han, C., Willett, P. K., and Abraham, D. A. (1999). Some methods to evaluate the performance of Page's test as used to detect transient signals. *Signal Processing, IEEE Transactions on*, 47(8):2112–2127.

Huang, Y., Cárdenas, A., Amin, S., Lin, Z., Tsai, H., and Sastry, S. (2009). Understanding the physical and economic consequences of attacks on control systems. *International Journal of Critical Infrastructure Protection*, 2(3):73–83.

Koch, K.-R. (1999). *Parameter Estimation and Hypothesis Testing in Linear Models (2nd ed.)*. Springer, Berlin, Heidelberg, DE.

Lai, T. L. (1995). Sequential changepoint detection in quality control and dynamical systems. *Journal of the Royal Statistical Society. Series B (Methodological)*, pages 613–658.

Lai, T. L. (1998). Information bounds and quick detection of parameter changes in stochastic systems. *Information Theory, IEEE Transactions on*, 44(7):2917–2929.

Lai, T. L. (2001). Sequential analysis: some classical problems and new challenges. *Statistica Sinica*, 11(2):303–350.

Lorden, G. (1971). Procedures for reacting to a change in distribution. *The Annals of Mathematical Statistics*, pages 1897–1908.

Moustakides, G. (1986). Optimal stopping times for detecting changes in distributions. *The Annals of Statistics*, 14(4):1379–1387.

Moustakides, G. V. (2014). Multiple Optimality Properties of the Shewhart Test. *Sequential Analysis*, 33(3):318–344.

Page, E. (1954). Continuous inspection schemes. *Biometrika*, pages 100–115.

Pasqualetti, F. (2012). *Secure Control Systems: A Control-Theoretic Approach to Cyber-Physical Security*. PhD thesis, University of California.

Pasqualetti, F., Dorfler, F., and Bullo, F. (2013). Attack detection and identification in cyber-physical systems. *Automatic Control, IEEE Transactions on*, 58(11):2715–2729.

Pollak, M. (1985). Optimal detection of a change in distribution. *The Annals of Statistics*, pages 206–227.

Polunchenko, A. and Tartakovsky, A. (2012). State-of-the-art in sequential change-point detection. *Methodology and computing in applied probability*, 14(3):649–684.

Poor, H. V. and Hadjiliadis, O. (2009). *Quickest Detection*. Cambridge, UK.

Premkumar, K., Kumar, A., and Veeravalli, V. (2010). Bayesian quickest transient change detection. In *Proc. Fifth International Workshop on Applied Probability (IWAP)*, pages 1–3.

Repin, V. (1991). Detection of a signal with unknown moments of appearance and disappearance. *Problemy Peredachi Informatsii*, 27(2):61–72.

Ritov, Y. (1990). Decision theoretic optimality of the cusum procedure. *The Annals of Statistics*, pages 1464–1469.

Roberts, S. (1966). A comparison of some control chart procedures. *Technometrics*, 8(3):411–430.

Shiryaev, A. (1963). On optimum methods in quickest detection problems. *Theory of Probability & Its Applications*, 8(1):22–46.

Smith, R. S. (2011). A decoupled feedback structure for covertly appropriating networked control systems. *Proc. IFAC World Congress*, pages 90–95.

Streit, R. L. and Willett, P. K. (1999). Detection of random transient signals via hyperparameter estimation. *Signal Processing, IEEE Transactions on*, 47(7):1823–1834.

Tartakovsky, A. (2005). Asymptotic performance of a multichart cusum test under false alarm probability constraint. In *Decision and Control, 2005 and 2005 European Control Conference. CDC-ECC'05. 44th IEEE Conference on*, pages 320–325. IEEE.

Tartakovsky, A. and Moustakides, G. (2010). State-of-the-art in bayesian changepoint detection. *Sequential Analysis*, 29(2):125–145.

Tartakovsky, A., Nikiforov, I., and Basseville, M. (2014). *Sequential Analysis : Hypothesis Testing and Changepoint Detection*. CRC Press, Taylor & Francis Group.

Willsky, A. and Jones, H. (1976). A generalized likelihood ratio approach to the detection and estimation of jumps in linear systems. *Automatic Control, IEEE Transactions on*, 21(1):108–112.

Zetter, K. (2011). H(ackers)2O : attack on city water station destroys pump. *Wired.com*. Online.