Measuring and modelling multivariate and spatial dependence of extremes

Titre: Quantifier et modéliser les dépendances multivariée et spatiale des extrêmes

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Abstract: Within both multivariate extreme models and spatial processes, it is important to precisely identify dependencies among extremes. In particular detecting asymptotic independence is fundamental and in a multivariate setting, many authors have proposed measures of extreme dependence/independence with sometimes associated tests. Dependence structures within spatial processes are more complex and very few authors have considered other structures than the max-stable one. In this survey paper some major contributions to inference and modelling for extremal dependencies are presented in multivariate and spatial contexts, including the presentation of recent spatial models allowing asymptotic independence.

Résumé : Lorsque l'on s'intéresse aux extrêmes multivariés ou spatiaux, il est important d'identifier au mieux la nature des dépendances entre réalisations extrêmes. La notion d'indépendance asymptotique des extrêmes est à ce titre primordiale. Dans un cadre multivarié, différentes mesures quantifiant la dépendance extrêmale ont été proposées et des tests d'indépendance asymptotique mis en oeuvre. Pour le cadre spatial, la modélisation classique s'appuie sur les processus max-stables mais les structures de dépendance présentes peuvent être plus complexes que celles gérées par ce type de processus. Dans cet article, les principaux résultats sur l'inférence et la modélisation de la dépendance entre réalisations extrêmes sont présentés et l'accent est mis sur les modèles récents permettant une prise en compte de l'indépendance asymptotique.

Keywords: Asymptotic dependence/independence, Multivariate extreme dependence, Spatial extreme dependence, Max-stable process, Inverse max-stable process
Mots-clés : Dépendance/Indépendance asymptotique, Dépendance extrême multivariée, Dépendance extrême spatiale, Processus max-stable, Processus inverse max-stable
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1. Introduction

In the last decades there have been significant advances in the available modelling procedures for univariate extreme values (see Gardes and Girard (2013) for details). Nervertheless, in practice, problems are often multivariate in character and more general questions can be addressed with multivariate models such as probability of at least one town being flooded in a hydrological example, for instance. Moreover, in many practical situations, it is more helpful to model the process of extreme levels as a continuous rather than a discrete process. It is particularly true when dealing with environmental extremes and max-stable processes introduced by de Haan (1984) allow a generalization of the multivariate extremes dependence structure to the infinite dimensional case. Within both multivariate extreme models and spatial processes, extremal dependencies play

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a fundamental role and tools for summarizing and exploring such dependencies are essential. The paper is organized as follows. Sections 2 and 3 deal with multivariate extremal dependence. Multivariate extreme value distributions are briefly introduced in Section 2, focusing on the bivariate case. Section 3 is devoted to summary measures for extremal dependence, including both notions of asymptotic dependence and asymptotic independence. The statistical estimation of the considered measures is discussed and the section ends with a review of recent statistical tests for asymptotic independence. Spatial processes are considered in Section 4. Max-stable processes, inverse max-stable processes and a mixture of both are successively considered. These three classes of processes correspond to models with asymptotic dependence within the max-stable dependence structures, asymptotic independence and asymptotic dependence apart from the max-stable dependence structures, respectively. A conclusion is given in Section 5.

2. Multivariate extreme value distributions

Let $(\mathbf{X_i})_{1 \le i \le n}$ be a sequence of n independent and identically distributed (iid) d-dimensional random vectors from a given distribution function. Multivariate extreme value theory relies on the study of the limiting behavior of the componentwise maxima $(\max_{1 \le i \le n} X_{i,1}, \ldots, \max_{1 \le i \le n} X_{i,d})$. As in the univariate case, the classes of Multivariate Extreme Value Distributions (MEVDs) and max-stable distributions actually coincide. Unfortunately, unlike the univariate case, the class of MEVDs does not admit any finite parametric representation. From max-stability, G is a MEVD if and only if (iff) for every positive integer n, there exist vectors $\mathbf{b}_n > 0$ and $\mathbf{a}_n$ such that $G^n(\mathbf{a}_n + \mathbf{b}_n \mathbf{x}) = G(\mathbf{x})$, $\mathbf{x} \in \mathbb{R}^d$. As a consequence, $G^{\frac{1}{n}}$ is a distribution function (df) for every positive integer n, that is G is max-infinitely divisible (Balkema and Resnick (1977)) which insures that there exists a measure μ on $[-\infty, \infty)$ such that $G(\mathbf{x}) = \exp(-\mu[-\infty, \mathbf{x}]^c)$. The measure μ, named the exponent measure, is unique if G is defined such that $G(\mathbf{x}) = 0$ for $\mathbf{x} \not\in]q_1, \infty] \times \ldots \times]q_d, \infty]$ where $q_j = \inf\{x \in \mathbb{R} : G_j(x) > 0\}$. From this property, MEVDs share a specific structure which can be used to characterization results. As showed in de Haan and Resnick (1977), there exist mainly two ways for characterizing a MEVD. In the sequel, Theorem 1 focuses on a particular one and, for sake of simplicity, corresponding results are expounded only in the bivariate case. The second one, based on a non-homogeneous Poisson process, will not be detailed in this review. More detailed probability and statistical accounts on MEVDs can be found in the following reference books: Resnick (1987); Beirlant et al. (2004); de Haan and Ferreira (2006).

Let $(X_i, Y_i)_{i=1,\ldots,n}$ be iid random vectors from a df F and denote by $\mathbf{M_n}$ the vector of componentwise maxima $\mathbf{M_n} = (M_{x,n}, M_{y,n}) = (\max_{1 \le i \le n} X_i, \max_{1 \le i \le n} Y_i)$.

If there exist two vector sequences of reals $\mathbf{a_n} = (a_{n,1}, a_{n,2})^t > 0$ and $\mathbf{b_n} = (b_{n,1}, b_{n,2})^t$ such that

$$\mathbb{P}(\mathbf{M_n} \le \mathbf{a_n}\mathbf{x} + \mathbf{b_n}) = F^n(\mathbf{a_n}\mathbf{x} + \mathbf{b_n}) \longrightarrow G(\mathbf{x}) \text{ as } n \to \infty$$

with $\mathbf{x} = (x_1, x_2)^t \in \mathbb{R}^2$ and G a non degenerate distribution, then G is a bivariate extreme value distribution and we say that F belongs to the domain of attraction of G denoted by $F \in D(G)$.

The df G is characterized both by its two margins and its dependence structure. From univariate extreme value theory, margins are Generalized Extreme Value (GEV) distributed. There is no loss of generality in assuming a particular GEV distribution (Resnick (1987)) and a common choice is to consider the unit Fréchet distribution $\Psi(y) = \exp(-y^{-1})$, $y > 0$. Such a choice allows

to isolate dependence aspects from marginal distributions. The normalising sequences are then usually taken as $a_{1,n} = a_{2,n} = n$, $b_{1,n} = b_{2,n} = 0$.

A possible characterization of the limiting joint distribution of componentwise maxima is presented in the following theorem (Coles (2001)).

Theorem 1. *Let $(X_i, Y_i)_{i=1,...,n}$ be iid random vectors from a df F having its marginal distributions in the domain of attraction of a unit Fréchet distribution. If*

$$\mathbb{P}\left(M_{x,n}/n \leq x_1, M_{y,n}/n \leq x_2\right) \longrightarrow G(x_1, x_2) \text{ as } n \to \infty$$

where G is a non degenerate distribution, then G is of the form :

$$G(x_1, x_2) = \exp\left(-V(x_1, x_2)\right), \quad x_1 > 0, x_2 > 0$$

with

$$V(x_1, x_2) = \int_0^1 \max\left(\frac{\omega}{x_1}, \frac{1-\omega}{x_2}\right) dH(\omega)$$

and H is a non-negative measure on $[0,1]$ satisfying the two following constraints:

$$\int_0^1 \omega \, dH(\omega) = \int_0^1 (1-\omega) \, dH(\omega) = 1.$$

The function V is related to the aforementioned exponent mesure μ defined on $[\mathbf{0}, \infty)\backslash\{\mathbf{0}\}$. In particular, V is a homogeneous function of order -1 that means $V(kx_1, kx_2) = \frac{1}{k}V(x_1, x_2)$. Using this property it is easy to check the max-stability property of G:

$$G^k(kx_1, kx_2) = \exp(-kV(kx_1, kx_2)) = \exp\left(-V(x_1, x_2)\right) = G(x_1, x_2).$$

Remark 1. If H is a measure which places mass 1 on $\omega = 0$ and on $\omega = 1$, then $G(x_1, x_2) = \exp(-x_1^{-1})\exp(-x_2^{-1})$, $x_1 > 0$, $x_2 > 0$ corresponding to the independent case. If H is a measure which places mass 2 on $\omega = 0.5$, $G(x_1, x_2) = \exp(-\max(x_1^{-1}, x_2^{-1}))$, $x_1 > 0$, $x_2 > 0$, corresponding to the perfect dependence, i.e. X=Y a.s..

Remark 2. As already stated, the theorem puts forward that finite parametric forms for G are unavailable. Nevertheless, in practice, it can be helpful to consider wide parametric families defining classes of bivariate extreme value distribution (e.g. the logistic, asymetric logistic, bilogistic families, see Coles (2001)).

Remark 3. In the bivariate case, another well-known representation for G has been proposed by Pickands (1981): G is a bivariate extreme value law with Fréchet margins iff there exists a function $A(\cdot)$ such that

$$G(x, y) = \exp\left(-\left(\frac{1}{x} + \frac{1}{y}\right)A\left(\frac{x}{x+y}\right)\right)$$

where $A(\cdot)$ is a convex function from $[0,1]$ to $[1/2,1]$ satisfying $\min(t, 1-t) \leq A(t) \leq 1$ for all $t \in [0,1]$, $A(0) = A(1) = 1$, $-1 \leq A'(0) \leq 0$, $0 \leq A'(1) \leq 1$ and $A''(t) \geq 0$. If $A(x) = \max(x, 1-x)$ (respectively $A(x) = 1$), then X and Y are perfectly dependent (resp. X and Y are independent).

The $A(\cdot)$ function is often called the *Pickands dependence function* and is related to the measure H through the relation

$$A(u) = \int_0^1 \max\left(u(1-\omega), (1-u)\omega\right) dH(\omega).$$

Remark 4. For a d-dimensional multivariate extreme value distribution G, the theorem states that there exists a finite measure H on the unit-simplex $S = \{\mathbf{x} \in [0,\infty]^d \setminus \mathbf{0} : \|\mathbf{x}\| = 1\}$ satisfying $\int_S w_j dH(w) = 1$ for all $j = 1, \ldots, d$ such that $G(\mathbf{x}) = \exp\left(-\int_S \max_{1 \leq j \leq d} \frac{w_j}{x_j} dH(\mathbf{w})\right)$.

3. Multivariate extreme dependence

From a theoretical point of view, the dependence structure of a d-dimensional extreme value distribution is completely characterized through the measure H. Nevertheless, H has a complex structure which cannot be easily inferred from data and simpler dependence measures are needed. Let $(X,Y) \sim F \in \mathscr{D}(G)$ with unit Fréchet margins. When G is equal to a product of its marginal unit Fréchet laws, that is when F is in the domain of attraction of independence, X and Y are said to be *asymptotically independent*. In other cases, X and Y are said to be *asymptotically dependent*. As an example, Sibuya (1960) showed that jointly normal variables which are not perfectly correlated are asymptotically independent. Clearly, independence implies asymptotic independence but the converse is false. Asymptotic independence is a rather complicated notion which has little to do with the amount of independence of the component of the original vector. As explained below, all max-stable distributions correspond either to an asymptotic dependence structure or an exact independence one. In other words, for max-stable models, the only achievable asymptotic independence structure is the usual independence notion. As emphasized by Coles et al. (1999), detecting asymptotic independence is fundamental. In particular, fitting asymptotic dependent models to asymptotically independent data leads to an over- or under-estimation of probabilities of extreme joint events, since there is a mis-placed assumption that the most extreme marginal events may occur simultaneously (Coles et al. (1999)). In the sequel, some usual extremal dependance measures are detailed in a bivariate extreme context.

3.1. *Measures of extreme dependence/independence*

3.1.1. *The upper tail dependence parameter χ and the $\bar{\chi}$ parameter*

In the bivariate case, Joe (1993) has introduced the *upper tail dependence* (UTD) parameter of a vector (X,Y) with distribution F and same marginal distribution F_X as $\lim_{x \to x_{F_X}} P(X > x \mid Y > x)$, where $x_{F_X} = \sup\{x \mid F_X(x) < 1\}$. If $F \in \mathscr{D}(G)$, then the UTD parameters associated to the distributions F and G coincide (Joe (1997)). Following Coles et al. (1999), we denote the UTD parameter $\chi \in [0,1]$. The χ parameter allows to discriminate between asymptotic independence (AI) and asymptotic dependence (AD) since AI (respectively AD) is achieved iff $\chi = 0$ (resp. $\chi > 0$) (Sibuya (1960)).

To work on extremal dependencies, Coles et al. (1999) introduce an equivalent formulation for χ which provides a possible diagnostic check for AD. To be as general as possible, the χ

parameter is expressed through (U, V) with $[0, 1]$-uniform marginals. Since we have,

$$P(V > u \mid U > u) = 2 - \frac{1 - P(U \leq u, V \leq u)}{1 - u} \sim 2 - \frac{\log P(U \leq u, V \leq u)}{\log(u)},$$

we deduce that

$$\chi = \lim_{u \to 1} 2 - \frac{\log P(U \leq u, V \leq u)}{\log P(U \leq u)} \equiv \lim_{u \to 1} \chi(u). \tag{1}$$

Consequently the function $\chi(u)$ can be interpreted as a quantile-dependent measure of dependence. More specifically, the sign of $\chi(u)$ indicates if variables are positively associated (that is probability of joint excesses is larger than under independence) or negatively associated at the quantile level u with $2 - \log(2u - 1)/\log(u) \leq \chi(u) \leq 1$ where the lower bound is interpreted as $-\infty$ for $u \leq \frac{1}{2}$, and 0 for $u = 1$. If G is a bivariate extreme value distribution, the function $\chi(u)$ is constant and it is easy to show that $\chi(u) = 2 - V(1, 1)$, leading to $\chi = 2 - V(1, 1)$, where $V(\cdot, \cdot)$ is defined in Theorem 1. Other particular cases are exact dependence and independence which respectively correspond to $\chi(u) = 1$ and $\chi(u) = 0 \ \forall u \in (0, 1]$.

Nevertheless, in more general cases, the $\chi(u)$ function appears as a non-trivial function. For example, for a Gaussian vector with correlation coefficient ρ, Coles et al. (1999) exhibit that $\chi(u)$ appears as linear from intermediate values of u whatever the ρ value is but with a diminished effect of the dependence as $u \to 1$: $\chi(u) \to 0$ for all $\rho < 1$. For $\rho > 0$, the convergence to 0 is very slow and appears as ultimately abrupt. That is, even though in presence of asymptotic independence, the $\chi(u)$ function could be interpreted as constant and positive. For asymptotically independent vectors, both χ parameter and $\chi(u)$ function appear of limited interest: the null value of χ does not provide any information on the strength of the dependence and the $\chi(u)$ function can be erroneously interpreted. For these reasons, Coles et al. (1999) proposed a new measure of dependence, named $\bar{\chi}$, which is relevant in the case of asymptotic independence. Using an analogy with the χ construction, they proposed to consider the parameter $\bar{\chi}$ defined as follows:

$$\bar{\chi} = \lim_{u \to 1} \bar{\chi}(u) \equiv \lim_{u \to 1} \frac{2 \log(1 - u)}{\log P(U > u, V > u)} - 1.$$

We have $-1 < \bar{\chi}(u) \leq 1$, $\forall u \in [0, 1]$, and $\bar{\chi}(u)$ increases in absolute value with the dependence. If $\bar{\chi}(u) \in]0, 1[$, extremes are positively associated whereas if $\bar{\chi}(u) \in]-1, 0[$, extremes are negatively associated.

The $\bar{\chi}$ parameter allows also to discriminate between AI and AD since AD (respectively AI) is achieved iff $\bar{\chi} = 1$ (resp. $\bar{\chi} < 1$). Moreover it characterizes the case of asymptotic independence. For instance, in the Gaussian case, the parameter $\bar{\chi}$ is equal to ρ and gives an information on the strength of the dependence even if we are in the case of asymptotic independence ($\rho < 1$). Finally, it is easy to show that $\bar{\chi}$ is equal to one for all bivariate max-stable distributions, apart from the independence one $G(x, y) = \exp(-\frac{1}{x}) \exp(-\frac{1}{y})$. In other words, the only asymptotic independence structure for max-stable distribution is the classical independence one.

3.1.2. The extremal coefficient θ and the coefficient of tail dependence η

Let (X, Y) be distributed according to a bivariate distribution F with common marginals F_X, $F \in \mathscr{D}(G)$, and consider $\theta_F(x) = \frac{\log P(X \leq x, Y \leq x)}{\log P(X \leq x)}$. This parameter, introduced by Buishand (1984),

is linked to the UTD parameter since $\chi = 2 - \lim_{x \to x_{F_X}} \theta_F(x)$ and verifies $F(x,x) = F_X(x)^{\theta_F(x)}$. The extremal index θ for a bivariate extreme value distribution G is defined as the limit of $\theta_G(x)$ when $x \to x_{G_X}$. With the usual Fréchet margins, we then have $G(x,x) = \exp\left(-\frac{\theta}{x}\right)$, $x > 0$. It is straightforward to deduce that $\chi = 2 - \theta$ and $\theta \in [1,2]$. Note that we also have $\theta = 2A\left(\frac{1}{2}\right)$ where $A(\cdot)$ is the Pickands dependence function introduced hereabove. The θ value summarizes the dependence of the margins and quantifies his strength: $\theta = 2$ corresponds to the independent case whereas $1 \le \theta < 2$ corresponds to asymptotic dependence; if $\theta = 1$, X and Y are perfectly dependent. Note that the extremal coefficient definition can easily be extended to the multivariate case ($d-$dimension) with $\theta \in [1,d]$. Properties attached to extremal coefficients are detailed in Schlather and Tawn (2002, 2003).

The extremal coefficient θ is of particular interest when dealing with asymptotic dependence but is useless in the case of AI. To resolve it, Ledford and Tawn (1996, 1997) introduced a new coefficient η called *coefficient of tail dependence*. Motivated by the expression of the joint survival function of unit Fréchet variables with bivariate normal dependence, they introduced the tail model

$$P(X > x, Y > x) = x^{-\frac{1}{\eta}} \mathscr{L}(x) \tag{2}$$

as $x \to \infty$ where $0 < \eta \le 1$ and $\mathscr{L}(\cdot)$ is a slowly varying function. The strength of the dependence is determinated through η and $\mathscr{L}(\cdot)$. The η parameter determines the decay rate of $1 - F(x,x) \equiv \bar{F}(x,x)$ for large x and characterizes the nature of the tail dependence. Four classes of dependence have been put forward (see Ledford and Tawn (1997) for details): asymptotic dependence, $\eta = 1$ and $\mathscr{L}(x) \nrightarrow 0$; negative association $0 < \eta < \frac{1}{2}$; positive association $\frac{1}{2} < \eta < 1$; near independence $\eta = \frac{1}{2}, \mathscr{L}(x) \ge 1$ (exact independence corresponds to $\mathscr{L}(x) \equiv 1$). The last three cases correspond to asymptotic independence since $\eta < 1$: in such cases, the η value hence measures the degree of dependence in the asymptotic independence. Note that Heffernan (2000) proposed evaluations of η for a large class of distributions.

Under the tail model assumption (2), Coles et al. (1999) showed that the $\bar{\chi}$ and η parameters are linked through the relation $\bar{\chi} = 2\eta - 1$.

3.1.3. *Dependence parameters estimation*

Empirical estimations of $\chi(u)$ and $\bar{\chi}(u)$ can always be deduced from empirical estimations of the bivariate survival distribution function. Coles et al. (1999) suggested to build confidence intervals assuming independence of the observations. Nevertheless they also argued that care has to be taken not to draw strong conclusions from these intervals but only consider these intervals as an informal picture of the extremal dependence, especially for u near 0 or 1.

Parametric inference for χ and $\bar{\chi}$ can be deduced using the Ledford and Tawn tail model (2): for (X,Y) with Fréchet margins and $T = \min(X,Y)$, the tail model leads to $P(T > x) \sim x^{-\frac{1}{\eta}} \mathscr{L}(x)$ as $x \to \infty$ and η appears as the shape parameter of T. Standard univariate techniques such as the Hill estimator can then be applied to estimation of η or equivalently of $\bar{\chi}$ (see e.g. Ledford and Tawn (1996); Peng (1999); Draisma et al. (2004) for details). A practical example for such inference procedures can be found for instance in Poon et al. (2003).

Smith (1990) was the first to propose an estimator for θ, using the fact that $1/\max(X,Y)$ is exponentially distributed with mean $\frac{1}{\theta}$ for (X,Y) with Fréchet marginal distributions. As already

stated, $\theta = 2A(\frac{1}{2})$ and non parametric estimators for the Pickands dependence function $A(\cdot)$ (Capéraà et al. (1997)) can also be used to estimate θ. Other estimators for θ have been proposed in a spatial context when considering pairs of sites separated by a distance h. Such estimators, also usable for non spatial bivariate vectors, are presented in Section 4.1.

3.2. *Existing tests for asymptotic independence*

There exists an important literature on testing the bivariate asymptotic independence (see de Carvalho and Ramos (2012) for a recent review). Our aim here is neither to give an exhaustive review nor to compare all the existing tests but to put forwards some useful approaches. A possible way to discriminate the raft of results is to distinguish between approaches related to a sample from the bivariate extreme value distribution (Tawn (1988); Ledford and Tawn (1996); Deheuvels and Martynov (1996); Ramos and Ledford (2005); Falk and Michel (2006); Bacro et al. (2010) and the references therein) and approaches which can be used on distributions in a domain of attraction (Zhang (2008); Hüsler and Deyuan (2009); Zhang et al. (2011)).
Let G a bivariate extreme value distribution (BEVD) with marginal distributions G_1 and G_2. In the sequel we focus on some asymptotic independence tests which are distribution free or based on joint tail modelling.

3.2.1. *Dependence function based test (Hüsler and Deyuan (2009))*

Let (X,Y) be a bivariate random vector with df $F = (F_1, F_2)$ and assume that $F \in \mathscr{D}(G)$, with $G_i(t) = \exp\left(-(1+\gamma_i x)^{-1/\gamma_i}\right)$, $\gamma_i \in \mathbb{R}$, $i = 1, 2$. From extreme value theory (see e.g. de Haan and Ferreira (2006)), G can be characterized through the so-called *dependence function* $\ell(\cdot, \cdot)$ defined as $\ell(x,y) = -\log G\left(\frac{x^{-\gamma_1}-1}{\gamma_1}, \frac{y^{-\gamma_2}-1}{\gamma_2}\right)$ on $[0,\infty]^2 \backslash (\infty, \infty)$. The dependence function is such that $\ell(x,y) = \lim_{t\to 0} t^{-1} P(1 - F_1(X) \le tx \text{ or } 1 - F_2(Y) \le ty)$ and satisfies $\min(x,y) \le \ell(x,y) \le x+y$, $x, y > 0$. If $\ell(x,y) = x+y$, for all $x, y > 0$, (X,Y) are asymptotically independent (see Beirlant et al. (2004)). Using this property, de Haan and de Ronde (1998) first proposed a graphical test for asymptotic independence. Hüsler and Deyuan (2009) develop two approaches to test the null hypothesis $H_0 : \ell(x,y) = x+y$. Both tests are based on the statistic $\ell_n(x,y) = \frac{m}{nk} \sum_{i=1}^n \mathbb{1}_{\{X_i \ge \tilde{X}_{m,m+1-[kx]} \text{ or } Y_i \ge \tilde{Y}_{m,m+1-[ky]}\}}$ where the sample $(X_1, Y_1), \ldots, (X_{n+m}, Y_{n+m})$ has been divided in two sub-samples (X_i, Y_i), $i = 1, \ldots, n$ and $(\tilde{X}_i, \tilde{Y}_i) = (X_{n+i}, Y_{n+i})$, $i = 1, \ldots, m$, with $m \equiv m(n)$ such that $m/n \to c > 0$ as $n \to \infty$ and $m/n - c = o(n^{-1/2})$. Here, as usual, $k \equiv k(n)$ designates an intermediate sequence such that $k(n) \to \infty$ as $n \to \infty$ and $k = o(n)$. Studying the weak convergence of the process $\sqrt{k}(\ell_n(x,y) - (x+y))$, $(x,y) \in [0,1]^2$ under H_0, Hüsler and Deyuan (2009) construct two tests which are similar to Cramer-von Mises and Kolmogorov-Smirnov tests respectively. In practice, if the sample size $m+n$ is even (resp. odd), possible choices for m and n are $m = n = \frac{n+m}{2}$ (resp. $m - 1 = n = \frac{n+m+1}{2}$) and several values for k are usually considered, for example $1 \le k \le n^{0.8}$, in order to appreciate how the k value could affect the results.

3.2.2. *Conditional distribution of sum based test (Falk and Michel (2006))*

Originally set up for bivariate distributions with reverse exponential margins, the test proposed by Falk and Michel (2006) can be adapted to GEV margins. More specifically, assume unit Fréchet

margins and consider $\varepsilon > 0$ and $t \in [0,1]$. When ε tends to 0, Falk and Michel (2006) show that the conditional distribution function $K_\varepsilon(t) \equiv P(X^{-1} + Y^{-1} < \varepsilon t \mid X^{-1} + Y^{-1} < \varepsilon)$ tends to t^2 if (X, Y) are asymptotically independent, and t otherwise. This result is then used to test the asymptotic independence using classical goodness-of-fit tests such as the Kolmogorov-Smirnov or the likelihood ratio as well as the chi-square test. Note that Frick et al. (2007) have proposed a generalization of the Falk and Michel's work, based on a second order differential expansion of the spectral decomposition of G. They focused on the case of a null hypothesis of tail dependence against a composite alternative representing the various degrees of tail independence.

3.2.3. Madogram based test (*Bacro et al. (2010)*)

Let (X, Y) be a random vector from G and consider the random variable $W = \frac{1}{2} \mid G_1(X) - G_2(Y) \mid$. Because $G_1(X)$ and $G_2(Y)$ are uniformly distributed on $[0, 1)$, their dependence relationships specify the distribution of W. If $X = Y$ a.s., then the distribution of W is a Dirac distribution at 0. If X and Y are independent, then W admits the probability df $f_W(z) = 4 - 8z$ on $[0, \frac{1}{2})$. Of course many intermediary situations are possible between perfect dependence and exact independence. In other words, the distribution of W provides information about the asymptotic dependence between X and Y. The mean and variance of W are respectively related to the extremal coefficient θ of (X, Y) by the formula $\mathbb{E}(W) = \frac{1}{2}\frac{\theta - 1}{\theta + 1}$ and $\sigma_W^2 = \frac{1}{6} - \frac{1}{4}\left(\frac{\theta - 1}{\theta + 1}\right)^2 - \frac{1}{2}\int_0^1 \frac{dt}{\left[1 + A(t)\right]^2}$ where $A(\cdot)$ is the classical Pickands dependence function. $\mathbb{E}(W)$ is a strictly monotonic increasing function of θ with $\mathbb{E}(W) = \frac{1}{6}$ if and only if $\theta = 2$. Accordingly, the asymptotic independence of X and Y can be checked by testing the null hypothesis $H_0 : \mathbb{E}(W) = \frac{1}{6}$ against $H_1 : \mathbb{E}(W) < \frac{1}{6}$. Under H_0, it can be shown that $\sqrt{n}\frac{\widehat{\mathbb{E}(W)} - \frac{1}{6}}{\widehat{\sigma_W}} \xrightarrow{\mathscr{D}} \mathscr{N}(0, 1)$ as $n \to \infty$. As the expectation $\mathbb{E}(W)$ can be related to the well-known madogram function used in spatial statistics, this test is referred to as the madogram test.

3.2.4. Tail quotient correlation based test (*Zhang (2008)*)

Let $(X_i, Y_i)_{1 \le i \le n}$ be a sequence of iid replications of a bivariate random vector (X, Y) with unit Fréchet margins and consider the *tail quotient correlation* introduced by Zhang (2008):

$$q_{u,n} = \frac{\max_{1 \le i \le n}\{(u + W_i)/(u + V_i)\} + \max_{1 \le i \le n}\{(u + V_i)/(u + W_i)\} - 2}{\max_{1 \le i \le n}\{(u + W_i)/(u + V_i)\}\max_{1 \le i \le n}\{(u + V_i)/(u + W_i)\} - 1}$$

where u is a positive threshold, W_i and V_i are exceedance values over u of X_i and Y_i respectively. Zhang (2008) proved that if X and Y are asymptotically independent, then $q_{u,n} \to 0$ a.s. as $n \to \infty$. In other words, a near zero value of $q_{u,n}$ would suggest a very weak tail dependence. Under the null hypothesis of asymptotic independence, Zhang (2008) establishes that $nq_{u,n} \xrightarrow{d} \Gamma(2, 1 - \exp(-1/u))$, with $\Gamma(\cdot, \cdot)$ the usual Gamma distribution. This result can be used to determine whether there is or not a tail dependence between X and Y. Note that as discussed in Zhang (2008), the truncation below the threshold value u of data does not make a rejection of the null hypothesis more likely.

3.2.5. *Joint tail censored likelihood based tests (Ramos and Ledford (2009))*

Let (X, Y) be a random vector with unit Fréchet margins. In a series of papers, Ledford and Tawn (1996, 1997) investigated the tail dependence of (X, Y) through a joint tail dependence model which can be summarized as $P(X > x, Y > y) = \frac{\mathscr{L}(x,y)}{x^{c_1}y^{c_2}}$, $c_1 + c_2 = 1/\eta$, where η is the aforementioned coefficient of tail dependence and $\mathscr{L}(\cdot, \cdot)$ is a bivariate slowly varying function with limit function $g(\cdot, \cdot)$ (Bingham et al. (1987)), that is

$$\lim_{t \to \infty} \frac{\mathscr{L}(tx, ty)}{\mathscr{L}(t, t)} = g(x, y), \quad \text{and} \quad g(cx, cy) = g(x, y) \text{ for all } c > 0, \ x > 0, y > 0.$$

The function g is constant along any ray $y = ax$ for $a > 0$ and a ray dependence function g_* defined as $g(x, y) = g_*(x/(x+y)) = g_*(w)$ for $w = x/(x+y) \in (0, 1)$ is introduced: g_* is termed ray dependent if $g_*(w)$ varies with w and ray independent if $g_*(w)$ is constant over differents rays. Recently, Ramos and Ledford (2009) considered a simplified version where $c_1 = c_2$, leading to the following model

$$P(X > x, Y > y) = \frac{\mathscr{L}(x, y)}{(xy)^{1/2\eta}}. \tag{3}$$

Such a model holds for a wide range of dependence models, spanning both asymptotic dependence and asymptotic independence (see Heffernan (2000)). Let u denote a high threshold. Ramos and Ledford main result is the following: if

$$\lim_{u \to \infty} P(X > us, Y > ut \mid X > u, Y > u) = \frac{g_*(s/(s+t))}{(st)^{1/2\eta}}, \tag{4}$$

for all $(s, t) \in [1, \infty] \times [1, \infty]$ then

$$\eta^{-1}g_*(w) = \left(\frac{1-w}{w}\right)^{1/2\eta} \int_0^w z^{1/\eta} dH_\eta(z) + \left(\frac{w}{1-w}\right)^{1/2\eta} \int_w^1 (1-z)^{1/\eta} dH_\eta(z) \tag{5}$$

where H_η is a non-negative measure on $[0, 1]$ satisfying the normalization condition

$$\frac{1}{\eta} = \int_0^{1/2} w^{1/\eta} dH_\eta(w) + \int_{1/2}^1 (1-w)^{1/\eta} dH_\eta(w). \tag{6}$$

Now, let (S, T) be a random vector with survival function $\bar{F}_{ST}$ such that $\bar{F}_{ST}$ can arise as a limit for a bivariate random vector (X, Y) with unit Fréchet margins (see Ramos and Ledford (2009) for details): $\bar{F}_{ST}(s, t) = P(S > s, T > t) = \lim_{u \to \infty} \frac{P(X > us, Y > ut)}{P(X > u, Y > u)}$. The following joint tail model can then be introduced: for a sufficiently large u, $P(X > x, Y > y)$ can be approximated by $P(X > u, Y > u)\bar{F}_{ST}(\frac{x}{u}, \frac{y}{u})$, $(x, y) \in (0, \infty)^2$, where from (4) and (5)

$$\bar{F}_{ST}(s, t) = \eta \int_0^1 \left\{ \min\left(\frac{w}{s}, \frac{1-w}{t}\right) \right\}^{1/\eta} dH_\eta(w).$$

Assuming a parametric model for H_η, likelihood based tests can be used for testing asymptotic dependence against asymptotic independence under the assumption that the joint tail model

holds exactly when $(x,y) > (u,u)$. Using the well-known threshold censored likelihood (Smith et al. (1997)), a (censored) likelihood $L_n(\eta, \lambda, \gamma)$ can be build from n iid vectors (X_i, Y_i), where γ denotes a vector of parameters related to the considered model and $\lambda = P(X > u, Y > u)$. Asymptotic dependence corresponds to $\eta = 1$, a boundary point of the parameter space which is non-regular with respect to the usual regularity conditions. Let $L_n(\hat{\eta}, \hat{\lambda}, \hat{\gamma})$ denote the maximum of the likelihood taken over all $\{\eta, \lambda, \gamma\}$ and $L_n(1, \tilde{\lambda}, \tilde{\gamma})$ the maximal likelihood under the constraint $\eta = 1$. Then, under the null hypothesis $\eta = 1$, $-2\log\left(L_n(1, \tilde{\lambda}, \tilde{\gamma})/L_n(\hat{\eta}, \hat{\lambda}, \hat{\gamma})\right)$ converges in distribution to V^2, where V is a non-negative random variable with distribution function $P(V \leq x) = \Phi(x) \mathbb{1}_{x \geq 0}$ for Φ the standard normal distribution function (Ramos and Ledford (2009)).

4. Dependence for spatial processes

4.1. Max-stable processes

In the last decade, there have been much interest in the statistical modelling of spatial extremes. One common way to deal with extreme value analysis in spatial statistics is by using max-stable processes which naturally extend multivariate extremes to spatial data (see e.g. Smith (1990); Coles (1993); Coles and Tawn (1996); Schlather (2002); de Haan and Pereira (2006); Cooley et al. (2006); Buishand et al. (2008); Kabluchko et al. (2009); Padoan et al. (2010); Lantuéjoul et al. (2011); Davison et al. (2012); Davison and Gholamrezaee (2012); Huser and Davison (2012) and the references therein). Max-stable processes are presented in Ribatet (2013) and below we will just remind the definition and an usually used spectral representation.

Definition 1. *Let $\mathscr{D}$ be an index space and consider a stochastic process $\{Z(s), s \in \mathscr{D}\}$ defined on $\mathscr{D}$. If there exist functions $a_n(\cdot) > 0$ and $b_n(\cdot)$ on $\mathbb{R}$ such that*

$$\max_{1 \leq i \leq n} \frac{Z_i(\cdot) - b_n(\cdot)}{a_n(\cdot)} \xrightarrow{d} Z(\cdot) \text{ as } n \to \infty$$

with $Z_1, Z_2, \ldots$ designate independent copies of Z, then $\{Z(s), s \in \mathscr{D}\}$ is a max-stable process.

A spectral representation for max-stable processes is the following (de Haan (1984); Schlather (2002)):

Proposition 1. *Assume that $\{r_i\}_{i \geq 1}$ are points of a Poisson process on $(0, \infty)$ with intensity dr. Let $\mathscr{D}$ be an index space and consider $\{W_i(s), s \in \mathscr{D}\}$, $i \geq 1$, be iid copies of a real valued random function on $\mathscr{D}$, independent of the $\{r_i\}$ and such that $E\{W^+(s)\} = \mu \in (0, \infty)$, where $W^+(s) = \max(W(s), 0)$. Then $Z(s) = \mu^{-1} \max_{i \geq 1} W_i^+(s)/r_i$ is a max-stable process with unit Fréchet margins on D.*

The Gaussian extreme value process (Smith (1990)), the extremal Gaussian process (Schlather (2002)) and the Brown-Resnick process (Kabluchko et al. (2009)) can be obtained by choosing particular expressions for $W_i(\cdot)$.

Characterising spatial dependence for an underlying max-stable processes is of fundamental interest for applications. For stationary processes, Schlather and Tawn (2003) introduced an extremal coefficient function $\theta(\cdot)$ to extend the extremal coefficient notion to spatial context. This function focuses on the extremal pairwise dependence: $\theta(h)$ measures the extremal dependence

of a pair of locations separated by a distance h. For a stationary spatial process, as emphasized by Schlather and Tawn (2003), the extremal coefficient function $\theta(\cdot)$ conveys sufficient information to characterize the amount of extremal dependence between pairs of variables separated by h. For a stationary max-stable process $Z(\cdot)$ with unit Fréchet marginal F, we have $P(Z(s) \leq z, Z(s+h) \leq z) = \exp(-\frac{\theta(h)}{z})$. For estimation of the extremal coefficient function, Schlather and Tawn (2003) proposed a maximum (censored) likelihood based estimator while Cooley et al. (2006) introduced geostatistic tools to completely characterize the extremal coefficient function. Using the so-called madogram function $v(h) = E|Z(s+h) - Z(s)|$, they defined the F-madogram as $v_F(h) = E|F(Z(s+h)) - F(Z(s))|$ and showed that

$$\theta(h) = \frac{1 + 2v_F(h)}{1 - 2v_F(h)}.$$

If $Z(\cdot)$ is assumed to be isotropic, it is then straightforward to derive from n independent replications of $Z(\cdot)$ an estimator for $\theta(h)$ using $\hat{v}_F(h) = \frac{1}{n}\sum_{k=1}^{n}\frac{1}{2K(h)}\sum_{(s,s')\in K(h)}|\hat{F}(Z(s+h)) - \hat{F}(Z(s))|$ where $K(h)$ denotes the number of pairs (s,s') separated by the distance h. A generalization of this approach has been proposed by Bel et al. (2008) who compared in a large simulation study four $\theta(\cdot)$- estimators, including the non-parametric one $\hat{\theta} = 2\hat{A}(\frac{1}{2})$ where $\hat{A}(\cdot)$ is the non-parametric estimator (Capéraà et al. (1997)) of the Pickands dependence function. Estimation of the F-madogram when the marginal distribution is unknown has been discussed by Naveau et al. (2009) who also further define the λ-madogram function by considering F^{λ} and $F^{1-\lambda}$, $0 < \lambda < 1$, into the F-madogram expression. This function is related to the Pickands dependence function and extends the notion of the madogram to completely describe the bivariate dependence structure.

As already stated, max-stable distributions model asymptotic dependence or exact independence. As a consequence, restrictive assumptions are imposed when using max-stable processes to model dependence for spatial extremes: a max-stable dependence structure is assumed and the only available asymptotic independence is the classical independence one. Moreover, the limiting model fitted at observable levels is assumed to be the true one for more extreme events than those which have been observed. As a consequence, situations where the extremal dependence structure varies with distance apart from max-stable dependence structure are rejected. Alternative models to max-stable ones are presented in the following subsection.

4.2. Asymptotically independent spatial processes

In a spatial context, Wadsworth and Tawn (2012) have proposed a flexible class of models adapted to the case of asymptotic independence. They construct this class by noting that an inverse max-stable process in the sense of the following proposition is asymptotically independent.

Proposition 2. *(Wadsworth and Tawn (2012)) Let $\{r_i\}$, $\{W_i^+(s)\}$, $i \geq 1$, $Z(\cdot)$ be as defined in Proposition 1. Then the process*

$$Y(s) = \mu \min_{i \geq 1} \frac{r_i}{W_i^+(s)} = \frac{1}{Z(s)}$$

with $\mathbb{P}(W^+(s) = W^+(s+h) > 0) = 0$, for all $||h|| > 0$ is an asymptotically independent process (AIP) on $\mathscr{D}$ with exponential margins.

In the same way that the extremal coefficient function $\theta(h)$ is defined as a summary of extremal dependence for a pair of locations separated by a distance h for max-stable processes, a coefficient of tail dependence function $\eta(h)$ based on the η coefficient for the AIPs can be defined (Ancona-Navarrete and Tawn (2002)).

For an inverse max-stable process $Y(\cdot)$, Wadsworth and Tawn (2012) noticed that the coefficient of tail dependence function $\eta(h)$ is equal to $1/\theta(h)$ with $\theta(h)$ the extremal coefficient function associated to the max-stable process $Z(\cdot)$.

As an illustration we present in Figure 1 two simulations of AIP processes. The first one on the left is constructed by inversing a Smith model with covariance parameters $\sigma_{11} = \sigma_{22} = 2$ and $\sigma_{12} = 0.75$. The second one is obtained by inversing a Schlather model with the powered exponential correlation function (sill=1, range=1 and smooth=1.5). The corresponding functions $\eta(h)$ are also represented. Keeping in mind that the case $\eta = 1/2$ corresponds to the near-independance case (Ledford and Tawn (1997)), the AIP constructed from a Smith process allows asymptotic independence but tends to near-independence for long distance. This is not true for an AIP constructed from a Schlather process which presents a stronger dependence in the independence when h is sufficiently large.
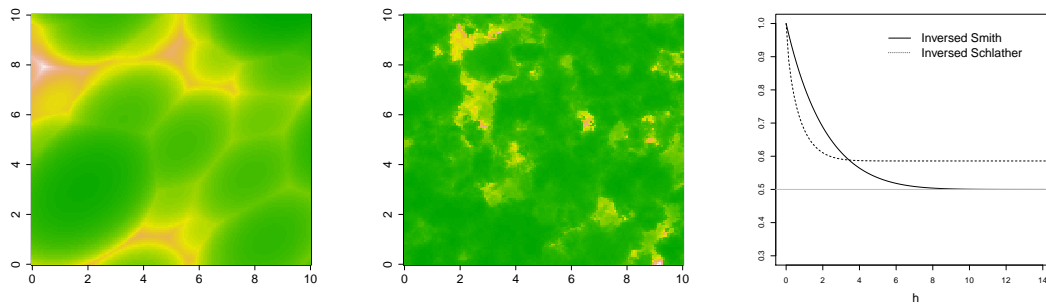

FIGURE 1. *On the left : simulation of an AIP constructed by inversing a Smith model. The central panel presents a simulation of an AIP constructed by inversing a Schlather model. On the right: associated functions $\eta(h)$.*

Inference for max-stable processes is possible using likelihood based on the pairwise densities. Padoan et al. (2010) have applied composite likelihood inference method (Lindsay (1988); Varin (2008)) to spatial extremes. Wadsworth and Tawn (2012) implemented the same technique for the inversed max-stable processes. To this aim, they need the bivariate distribution which is easy to compute with Fréchet margins and which obviously depends on the bivariate distribution of the corresponding max-stable process $Z(\cdot)$.

The composite likelihood information criterion named CLIC (Varin and Paolo (2005)) is helpful for the model selection when we deal with composite likelihood. The lower the CLIC, the better the fit. It is equal to $-2[L_p(\widehat{\beta}, x) - tr(\mathscr{J}(\widehat{\beta})\mathscr{H}(\widehat{\beta})^{-1}]$ with L_p the pairwise log-likelihood, $\widehat{\beta}$ the pairwise maximum likelihood estimate of β, $J(\widehat{\beta})$ the estimated variance matrix of the score vector and $H(\widehat{\beta})$ the hessian matrix of L_p.

To summarize, AIPs present an alternative to max-stable processes for data which present dependence at observed levels but independence in the limit. Obviously the problem we have in the bivariate case persists in the spatial one. It is not an easy task to decide in which case we are.

Have we to use a dependent or independent process ? Generally we observe dependence and it is difficult to know whether this dependence will persist in the limit or not.

It is essential to have theoretical results and practical tools which are adapted to both AD and AI cases.

In a recent paper, Guillou et al. (2013) adapt the madogram to the context of asymptotic independence. Their approach is based on results proposed by Ramos and Ledford (2011). Ramos and Ledford (2011) extend the classical and limited componentwise maxima result given in Theorem 1 by proposing an analogue that holds for both AD and AI cases. Under the same setup as subsection 3.2.5, we consider a sequence of iid random vectors (X_i, Y_i), $i = 1, \ldots, n$, distributed with the same distribution than (X, Y). Then defining $M_{.,n,\varepsilon b_n}$ as the componentwise maxima if we only consider vectors (X_i, Y_i) such that X_i and Y_i are greater than εb_n with b_n such that $nP(X > b_n, Y > b_n) = 1$, Ramos and Ledford (2011) obtained that

$$G_\eta(x,y) \equiv \lim_{\varepsilon \to 0} \lim_{n \to \infty} P\left(\frac{M_{X,n,\varepsilon b_n}}{b_n} \leq x, \frac{M_{Y,n,\varepsilon b_n}}{b_n} \leq y \right) = \exp(-V_\eta(x,y))$$

where

$$V_\eta(x,y) = \eta \int_0^1 \left(\max\left(\frac{\omega}{x}, \frac{1-\omega}{y} \right) \right)^{1/\eta} dH_\eta(\omega).$$

The margins are Fréchet distributed with shape parameter η and different scale parameters $\sigma_X = V_\eta(1, \infty)$ and $\sigma_Y = V_\eta(\infty, 1)$.

Guillou et al. (2013) defined the F-madogram v_η for a bivariate vector following the df $G_\eta(x,y)$ and proposed an analogue of the classical extremal coefficient by defining $\theta_\eta = \frac{1+2v_\eta}{1-2v_\eta}$ with $\theta_\eta = V_\eta(\sigma_X^\eta, \sigma_Y^\eta)$. By this way they generalized the F-madogram by allowing the pairwise maxima to belong to a wider family. Moreover they also proposed estimators of these quantities and applied them on simulated and real data. This approach is promising and should be also useful in a spatial context.

Coming back to spatial processes, Wadsworth and Tawn (2012) proposed also an hybrid spatial dependence model with the aim to allow a mixture of AI and AD to be present. This hybrid model is discussed in the next subsection.

4.3. Mixing max-stable and asymptotically independent models

Wadsworth and Tawn (2012) proposed to consider an hybrid spatial process $H(\cdot)$. Let $Z(\cdot)$ and $Z_1(\cdot)$ denote two max-stable processes and $Y(\cdot) = 1/Z_1(\cdot)$ an AIP each with Fréchet margins. The hybrid spatial process is defined as $H(s) = \max\{aZ(s), (1-a)Y(s)\}$ with $a \in [0,1]$.

Figure 2 presents simulation results for this process $H(\cdot)$. In these simulations we have supposed that the max-stable processes $Z(\cdot)$ and $Z_1(\cdot)$ belong to the same family with the same parameters used in Figure 1. The first column corresponds to the case $a = 0.2$, the second one to $a = 0.5$ and the last one to $a = 0.8$.

To exemplify the dependence structure of this spatial process, it is possible to compute

$$P(H(x) > z, H(x+h) > z) = \frac{a(2-\theta(h))}{z} + \frac{(1-a)^{1/\eta(h)}}{z^{1/\eta(h)}} + \mathcal{O}(z^{-2}) \quad (z \to +\infty)$$
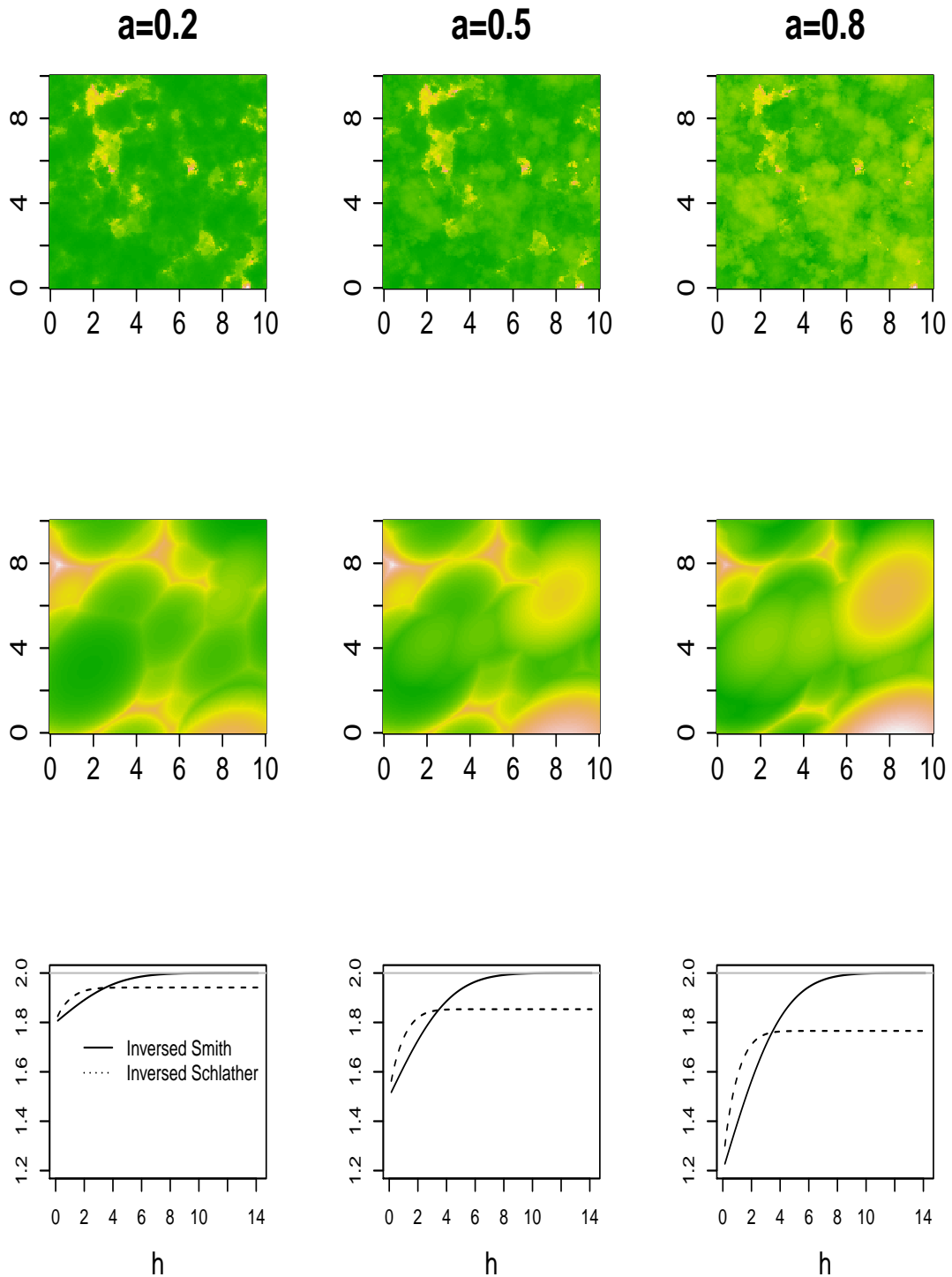
FIGURE 2. *Simulations of the process* $H(\cdot)$ *for* $a \in \{0.2, 0.5, 0.8\}$. *Smith process (resp. Schlather process) is involved in the first (resp. second line). Associated* $\theta(\cdot)$ *functions are presented in the third line.*

with $\theta(h)$ the extremal dependence function associated to $Z(\cdot)$ and $\eta(h)$ the coefficient of tail dependence function associated to $Y(\cdot)$. Based on this result, we can compute the parameter χ_H associated to the process $H(\cdot)$ for a distance h and we obtain $\chi_H(h) = a\,\chi_Z(h)$.

Obviously, this model includes the max-stable process (when $a = 1$) and the AIP one (when $a = 0$). In the case $a \in (0,1)$ the model is neither max-stable nor AI. Figure 2 illustrates how the spatial extremal dependencies vary regarding to the value of a. This new model extends traditional dependence modelling within the asymptotically dependent class and is convenient when AD is present at all distances because it allows to capture a second order in the dependence structure which is not possible with a max-stable model.

Concerning inference issues, since it is easy to compute the bivariate distribution associated to this hybrid process, estimation of the unknown parameters using the likelihood based on the pairwise densities is possible and the CLIC is used for model selection purposes.

5. Conclusion

This paper focused on extremal dependence in multivariate and spatial models, especially on asymptotic independence. Recent results on statistical inference and statistical modelling in relation to the asymptotic independence setting have been put forward.

For multivariate models, there exists a huge literature on such topics and other existing approaches based on point processes, spectral measures or conditional distributions given that at least one component is large (Heffernan and Tawn (2004)) for instance could also be considered. The study of spatial extreme processes is more recent and major efforts have been concentrated on developing models and methods taking into account the more complex structures of spatial phenomena. Innovative and original developments are needed to better tackle practical problems such as environmental or climatic ones. In particular, measuring and modelling extremal dependencies for spatio-temporal processes clearly appear as a new challenge for future research.

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