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Minimum distance estimators of the Pickands dependence function and related tests of multivariate extreme-value dependence

Titre: Estimateurs du minimum de distance de la fonction de dépendance de Pickands et tests associés d'appartenance à la classe des copules de valeurs extrêmes

Betina Berghaus [1] , Axel Bücher [1] and Holger Dette [1]

Abstract: We consider the problem of estimating the Pickands dependence function corresponding to a multivariate extreme-value distribution. A minimum distance estimator is proposed which is based on an L^2-distance between the logarithms of the empirical and the unknown extreme-value copula. The minimizer can be expressed explicitly as a linear functional of the logarithm of the empirical copula and weak convergence of the corresponding process on the simplex is proved. In contrast to other procedures which have recently been proposed in the literature for the nonparametric estimation of a multivariate Pickands dependence function (see [Zhang et al., 2008] and [Gudendorf and Segers, 2011]), the estimators constructed in this paper do not require knowledge of the marginal distributions and are an alternative to the method which has recently been suggested in [Gudendorf and Segers, 2012]. Moreover, the minimum distance approach allows the construction of a simple test for the hypothesis of a multivariate extreme-value copula, which is consistent against a broad class of alternatives. The finite-sample properties of the estimator and a multiplier bootstrap version of the test are investigated by means of a simulation study.

Résumé : Nous nous intéressons à l'estimation de la fonction de dépendance de Pickands correspondant à une distribution de valeurs extrêmes multivariée. Un estimateur du minimum de distance fondé sur la distance L^2 entre les logarithmes de la copule empirique et de la copule inconnue est proposé et sa convergence faible est démontrée. Contrairement à d'autres procédures récemment proposées dans la littérature pour l'estimation de la fonction de dépendance de Pickands multivariée (voir [Zhang et al., 2008] et [Gudendorf and Segers, 2011]), les estimateurs étudiés dans ce travail ne requièrent pas la donnée des distributions marginales et sont ainsi une alternative à la méthode de [Gudendorf and Segers, 2012]. De plus, l'approche du minimum de distance considérée permet naturellement la construction d'un test d'appartenance à la classe des copules de valeurs extrêmes dont la consistance est démontrée pour les copules modélisant une association positive. Des simulations sont enfin utilisées pour étudier empiriquement, sur des échantillons de taille finie, les propriétés de l'estimateur du minimum de distance ainsi que du test associé mis en oeuvre à l'aide d'un rééchantillonnage fondé sur des multiplicateurs.

Keywords: Extreme-value copula, minimum distance estimation, Pickands dependence function, weak convergence, empirical copula process
Mots-clés : copules de valeurs extrêmes, estimateurs du minimum de distance, fonction de dépendance de Pickands, convergence faible, processus de copule empirique
AMS 2000 subject classifications: 62G05, 62G32, 62G20

1. Introduction

Consider a d-dimensional random variable $\boldsymbol{X} = (X_1, \ldots, X_d)$ with continuous marginal distribution functions $F_1, \ldots, F_d$. It is well known that the dependency between the different components of $\boldsymbol{X}$

[1] Fakultät für Mathematik, Ruhr-Universität Bochum, 44780 Bochum, Germany.
E-Mail: betina.berghaus@rub.de, axel.buecher@rub.de and holger.dette@rub.de

can be described in a margin-free way by the copula C, which is based on the representation

$$F(x_1,\ldots,x_d) = C(F_1(x_1),\ldots F(x_d))$$

of the joint distribution function F of the random vector $\boldsymbol{X}$ (see [Sklar, 1959]). A prominent class of copulas is the class of extreme-value copulas which arise naturally as the possible limits of copulas of component-wise maxima of independent, identically distributed or strongly mixing stationary sequences (see [Deheuvels, 1984] and [Hsing, 1989]). For some applications of extreme-value copulas we refer to [Tawn, 1988, Ghoudi et al., 1998, Coles et al., 1999] or [Cebrian et al., 2003], among others.

A (d-dimensional) copula C is an extreme-value copula if and only if there exists a copula, say $\bar{C}$, such that the relation

$$\lim_{n\to\infty} \bar{C}(u_1^{1/n},\ldots,u_d^{1/n})^n = C(u_1,\ldots,u_d) \tag{1.1}$$

holds for all $\boldsymbol{u} = (u_1,\ldots,u_d) \in [0,1]^d$. Passing to a continuous limit, one can easily see that C is an extreme-value copula if and only if C is max-stable, i.e., the condition

$$\left\{C\left(u_1^{1/r},\ldots,u_d^{1/r}\right)\right\}^r = C(u_1,\ldots,u_d)$$

holds for all $\boldsymbol{u} \in [0,1]^d$ and all $r > 0$. There exists an alternative description of multivariate extreme-value copulas, which is based on a function on the simplex

$$\Delta_{d-1} := \left\{\boldsymbol{t} = (t_1,\ldots,t_{d-1}) \in [0,1]^{d-1} \,\middle|\, \sum_{j=1}^{d-1} t_j \le 1\right\}.$$

To be precise, a copula C is an extreme-value copula if and only if there exists a function

$$A : \Delta_{d-1} \to [1/d, 1]$$

such that C has a representation of the form

$$C(u_1,\ldots,u_d) = \exp\left\{\left(\sum_{j=1}^{d} \log u_j\right) A\left(\frac{\log u_2}{\sum_{j=1}^{d}\log u_j},\ldots,\frac{\log u_d}{\sum_{j=1}^{d}\log u_j}\right)\right\}. \tag{1.2}$$

The function A is called a Pickands dependence function (see [Pickands, 1981]). If relation (1.2) holds true then the corresponding Pickands dependence function A is necessarily convex and satisfies the inequalities

$$\max\left\{1 - \sum_{j=1}^{d-1} t_j, t_1,\ldots,t_{d-1}\right\} \le A(\boldsymbol{t}) \le 1 \tag{1.3}$$

for all $\boldsymbol{t} = (t_1,\ldots,t_{d-1}) \in \Delta_{d-1}$. In the case $d = 2$ these conditions are also sufficient for A to be a Pickands dependence function. By the representation (1.2) of the extreme-value copula C the problem of estimating C reduces to the estimation of the $(d-1)$-dimensional function A

and statistical inference for an extreme-value copula C may now be reduced to inference for its corresponding Pickands dependence function.

The problem of estimating the Pickands dependence function nonparametrically has a long history. Early work dates back to Pickands [Pickands, 1981] and Deheuvels [Deheuvels, 1991]. Alternative estimators have been proposed and investigated in [Capéraà et al., 1997, Rojo Jiménez et al., 2001, Hall and Tajvidi, 2000, Segers, 2007]. The corresponding authors discuss the estimation of the Pickands dependence function in the bivariate case and assume knowledge of the marginal distributions. Recently, Genest and Segers [Genest and Segers, 2009] and Bücher et al. [Bücher et al., 2011] proposed new estimators in the two-dimensional case which do not require this knowledge. While [Genest and Segers, 2009] considered rank-based versions of the estimators of [Pickands, 1981] and [Capéraà et al., 1997], the approach in [Bücher et al., 2011] is based on the minimum distance principle and yields an infinite class of estimators.

The estimation problem of the Pickands dependence function in the case $d > 2$ was studied in [Zhang et al., 2008] and [Gudendorf and Segers, 2011] assuming knowledge of the marginal distributions. Their estimators are based on functionals of the transformed random variables $Y_{ij} = -\log F_j(X_{ij})$ $(i = 1,\dots,n, j = 1,\dots,d)$, which were also the basis for the estimators proposed in [Pickands, 1981] and [Capéraà et al., 1997] in the bivariate case. Zhang, Wells and Peng [Zhang et al., 2008] considered the random variables

$$Z_{ij}(\boldsymbol{s}) = \frac{\bigwedge_{k:k\neq j} \frac{Y_{ik}}{s_k}}{\frac{Y_{ij}}{1-s_j} + \bigwedge_{k:k\neq j} \frac{Y_{ik}}{s_k}}$$

where $\boldsymbol{s} = (s_1,\dots,s_d) \in (0,1)\times\Delta_{d-1}$, $s_1 = 1-\sum_{j=2}^d s_j$ and $\bigwedge_{j\in\mathscr{J}} a_j = \min\{a_j \mid j \in \mathscr{J}\}$. They showed that the corresponding distribution function depends in a simple way on a partial derivative of the logarithm of the Pickands dependence function and proposed to estimate the Pickands dependence function by using a functional of the empirical distribution function of the random variables $Z_{ij}(\boldsymbol{s})$. The obtained estimator is uniformly consistent and converges point-wise to a normally distributed random variable.

Gudendorf and Segers [Gudendorf and Segers, 2011] discussed the random variable $\xi_i(\boldsymbol{s}) = \bigwedge_{j=1}^d \frac{Y_{ij}}{s_j}$ which is Gumbel-distributed with location parameter $\log A(s_2,\dots,s_d)$. They suggested to estimate the Pickands dependence function by the method-of-moments and also provided an endpoint correction to impose (some of) the properties of the Pickands dependence function. They also discussed the asymptotic properties of the estimator and a way to get optimal weight functions needed in the endpoint corrections. It was shown that the least squares estimator leads to weight functions which minimize the asymptotic variance. Furthermore, Gudendorf and Segers [Gudendorf and Segers, 2011] showed that in some cases their estimator coincides with the one proposed in [Zhang et al., 2008]. An extension of the methodology in [Gudendorf and Segers, 2011] to the case of unknown marginals has recently been considered in [Gudendorf and Segers, 2012].

The present paper is devoted to the construction of an alternative class of estimators of the Pickands dependence function in the general multivariate case $d \geq 2$, which also do not require knowledge of the marginal distribution. For this purpose we will use the minimum distance approach proposed in [Bücher et al., 2011], which allows us to construct an infinite dimensional class of estimators, which depend in a linear way on the logarithm of the d-dimensional empirical copula. Because this statistic does not require knowledge of the marginals the resulting estimator of the Pickands dependence function does automatically not depend on the marginal distributions of X. We also briefly discuss the properties of our methods in the case of dependent data.

Moreover, the minimum distance approach also allows us to construct a simple test for the hypothesis that a given copula is an extreme-value copula. In this case the distance between the copula and its best approximation by an extreme-value copula is 0, and as a consequence a consistent estimator of the minimum distance should be small. Therefore the hypothesis of an extreme-value copula can be rejected for large values of this estimator. A multiplier bootstrap for the approximation of the critical values is proposed and its consistency is proved. Moreover, we demonstrate that the new bootstrap is also applicable in the context of dependent data. Alternative tests for extreme-value dependence in dimension $d > 2$ in the case of independent data have recently been proposed in [Kojadinovic et al., 2011] (for tests in dimension $d = 2$ for independent data see, e.g., [Ghoudi et al., 1998, Ben Ghorbal et al., 2009, Kojadinovic and Yan, 2010b, Bücher et al., 2011, Genest et al., 2011, Quessy, 2012, Du and Nešlehová, 2012]).

The remaining part of the paper is organized as follows. In Section 2 we present the necessary notation and define the class of minimum distance estimators. The main asymptotic properties are given in Section 3, while the corresponding test for the hypothesis of an extreme value copula is investigated in Section 4. Here we also establish consistency of the multiplier bootstrap such that critical values can easily be calculated by numerical simulation. The finite-sample properties of the new estimators and the test are investigated in Section 5, where we also present a brief comparison with the estimators proposed in [Gudendorf and Segers, 2012] and an illustration of the test on the well-known uranium exploration data of Cook and Johnson [Cook and Johnson, 1986]. Finally, some technical details are deferred to an Appendix in Section 6.

2. Measuring deviations from an extreme-value copula

Throughout this paper we define $\mathscr{A}$ as the set of all functions $A : \Delta_{d-1} \to [1/d, 1]$ and Π is the independence copula, that is $\Pi(u_1, \ldots, u_d) = \prod_{j=1}^d u_j$. For most statements in this paper we will assume that the copula C satisfies $C \geq \Pi$ (or a slight modification of this statement). This assumption is equivalent to positive quadrant dependence of the random variables, that is for every $(x_1, \ldots, x_d) \in \mathbb{R}^d$ we have

$$\mathbb{P}(X_1 \leq x_1, \ldots, X_d \leq x_d) \geq \prod_{j=1}^{d} \mathbb{P}(X_j \leq x_j).$$

Obviously it holds for any extreme-value copula because of the lower bound of the Pickands dependence function. Following Bücher et al. [Bücher et al., 2011], the construction of minimum distance estimators for the Pickands dependence function is based on a weighted L^2-distance

$$M_h(C, A) = \int_{(0,1)\times\Delta_{d-1}} \left\{ \log C\left(y^{1-t_1-\ldots-t_{d-1}}, y^{t_1}, \ldots, y^{t_{d-1}}\right) - \log(y) A(\boldsymbol{t}) \right\}^2 h(y)\, d(y, \boldsymbol{t}), \quad (2.1)$$

where $h : [0,1] \to \mathbb{R}^+$ is a continuous weight function and $\boldsymbol{t} = (t_1, \ldots, t_{d-1}) \in \Delta_{d-1}$. The result below gives an explicit expression of the best L^2-approximation of the logarithm of a copula satisfying this condition.

Theorem 2.1. *Assume that the copula C satisfies $C \geq \Pi^\kappa$ for some $\kappa \geq 1$ and that the weight function h satisfies $\int_0^1 (\log y)^2 h(y)\,dy < \infty$. Then*

$$A^* = \operatorname{argmin}\{M_h(C,A) \mid A \in \mathscr{A}\}$$

is well-defined and given by

$$A^*(\boldsymbol{t}) = B_h^{-1} \int_0^1 \frac{\log C\left(y^{1-t_1-\dots-t_{d-1}}, y^{t_1}, \dots, y^{t_{d-1}}\right)}{\log y} h^*(y)\,dy, \tag{2.2}$$

where we use the notations

$$h^*(y) = \log^2(y)\,h(y) \tag{2.3}$$

and $B_h = \int_0^1 (\log y)^2 h(y)\,dy = \int_0^1 h^*(y)\,dy$. Moreover, if $C \geq \Pi$, the function A^* defined in (2.2) satisfies

$$\max\Big\{1 - \sum_{j=1}^{d-1} t_j, t_1, \dots, t_{d-1}\Big\} \leq A^*(\boldsymbol{t}) \leq 1.$$

Proof. Since $C \geq \Pi^\kappa$ we obtain

$$1 \geq C\left(y^{1-t_1-\dots-t_{d-1}}, y^{t_1}, \dots, y^{t_{d-1}}\right) \geq \Pi\left(y^{1-t_1-\dots-t_{d-1}}, y^{t_1}, \dots, y^{t_{d-1}}\right)^\kappa = y^\kappa$$

and thus

$$0 \geq \log\left(C\left(y^{1-t_1-\dots-t_{d-1}}, y^{t_1}, \dots, y^{t_{d-1}}\right)\right) \geq \kappa \log y.$$

This yields $|\log C\left(y^{1-t_1-\dots-t_{d-1}}, y^{t_1}, \dots, y^{t_{d-1}}\right)| \leq \kappa |\log y|$ and therefore the integral in (2.2) exists. By Fubini's theorem the weighted L^2-distance can be rewritten as

$$M_h(C,A) = \int_{\Delta_{d-1}} \int_0^1 \left(\frac{\log C\left(y^{1-t_1-\dots-t_{d-1}}, y^{t_1}, \dots, y^{t_{d-1}}\right)}{\log y} - A(\boldsymbol{t})\right)^2 \log^2(y)\,h(y)\,dy\,d\boldsymbol{t},$$

and now the first part of the assertion is obvious.

For a proof of the second part we make use of the upper Fréchet-Hoeffding-bound and obtain

$$\begin{aligned} A^*(\boldsymbol{t}) &\geq B_h^{-1} \int_0^1 \frac{\log \min\{y^{1-t_1-\dots-t_{d-1}}, y^{t_1}, \dots, y^{t_{d-1}}\}}{\log y} h^*(y)\,dy \\ &= B_h^{-1} \int_0^1 \max\Big\{1 - \sum_{j=1}^{d-1} t_j, t_1, \dots, t_{d-1}\Big\} h^*(y)\,dy = \max\Big\{1 - \sum_{j=1}^{d-1} t_j, t_1, \dots, t_{d-1}\Big\}. \end{aligned}$$

With a similar calculation and the assumption $C \geq \Pi$ we obtain the upper bound. □

A possible choice for the weight function is given by $h(y) = -y^k/\log y$, where $k \geq 0$, see Example 2.5 in [Bücher et al., 2011]. In Section 5 we consider this weight function with $k = 0.5$.

If the copula C is not an extreme-value copula the function A^* has not necessarily to be convex for any copula satisfying $C \geq \Pi$ (see [Bücher et al., 2011]). However, for every copula satisfying $C \geq \Pi^\kappa$ for some $\kappa \geq 1$ the equality $M_h(C,A^*) = 0$ holds if and only if the copula C is an extreme-value copula with the Pickands dependence function A^*. This property will be useful for the construction of a test for the hypothesis that C is an extreme-value copula, which will be discussed in Section 4. For this purpose we will need an empirical analogue of the "best approximation" A^* which is constructed and investigated in the following section.

3. Weak convergence of minimal distance estimators

Throughout the remaining part of this paper let $\boldsymbol{X}_1,\ldots,\boldsymbol{X}_n$ denote independent identically distributed $\mathbb{R}^d$-valued random variables. We define the components of each observation by $\boldsymbol{X}_i = (X_{i1},\ldots,X_{id})$ $(i = 1,\ldots,n)$ and assume that all marginal distribution functions of $\boldsymbol{X}_i$ are continuous. The copula of $\boldsymbol{X}_i$ can easily be estimated in a nonparametric way by the empirical copula (see, e.g., [Rüschendorf, 1976]) which is defined for $\boldsymbol{u} = (u_1,\ldots,u_d)$ by

$$C_n(\boldsymbol{u}) = \frac{1}{n}\sum_{i=1}^{n}\mathbb{I}\{\hat{U}_{i1} \le u_1,\ldots\hat{U}_{id} \le u_d\}, \tag{3.1}$$

where $\hat{U}_{ij} = \frac{1}{n+1}\sum_{k=1}^{n}\mathbb{I}\{X_{kj} \le X_{ij}\}$ denote the normalized ranks of X_{ij} amongst $X_{1j},\ldots,X_{nj}$. Following Bücher et al. [Bücher et al., 2011], we use Theorem 2.1 to construct an infinite class of estimators for the Pickands dependence function by replacing the unknown copula with the empirical copula. To avoid zero in the logarithm we use a modification of the empirical copula. We set $\tilde{C}_n = C_n \vee n^{-\gamma}$ where $\gamma > \frac{1}{2}$ and obtain the estimator

$$\widehat{A}_{n,h}(\boldsymbol{t}) = B_h^{-1}\int_0^1 \frac{\log\tilde{C}_n\left(y^{1-t_1-\ldots-t_{d-1}}, y^{t_1},\ldots,y^{t_{d-1}}\right)}{\log y} h^*(y)\,dy. \tag{3.2}$$

Weak convergence, denoted by the arrow $\rightsquigarrow$ throughout this paper, of the empirical process $\sqrt{n}(C_n - C)$ was investigated in [Rüschendorf, 1976] and [Fermanian et al., 2004], among others, under various assumptions on the partial derivatives of the copula C. Recently, Segers [Segers, 2012] proved the weak convergence of the empirical copula process

$$\mathbb{C}_n = \sqrt{n}\,(C_n - C) \rightsquigarrow \mathbb{G}_C \qquad \text{in } \left(\ell^\infty[0,1]^d, \|\cdot\|_\infty\right) \tag{3.3}$$

under a rather weak assumption, which is satisfied for many commonly used copulas, that is

$$\partial_j C(\boldsymbol{u}) \text{ exists and is continuous on } \{\boldsymbol{u} \in [0,1]^d \mid u_j \in (0,1)\} \tag{3.4}$$

for every $j = 1,\ldots,d$. The limiting process $\mathbb{G}_C$ in (3.3) depends on the unknown copula and is given by

$$\mathbb{G}_C(\boldsymbol{u}) = \mathbb{B}_C(\boldsymbol{u}) - \sum_{j=1}^{d}\partial_j C(\boldsymbol{u})\,\mathbb{B}_C(1,\ldots,1,u_j,1,\ldots,1), \tag{3.5}$$

where we set $\partial_j C(\boldsymbol{u}) = 0$, $j = 1,\ldots,d$ for the boundary points $\{\boldsymbol{u} \in [0,1]^d \mid u_j \in \{0,1\}\}$. Here, $\mathbb{B}_C$ is a centered Gaussian field on $[0,1]^d$ with covariance structure

$$r(\boldsymbol{u},\boldsymbol{v}) = \operatorname{Cov}(\mathbb{B}_C(\boldsymbol{u}), \mathbb{B}_C(\boldsymbol{v})) = C(\boldsymbol{u}\wedge\boldsymbol{v}) - C(\boldsymbol{u})C(\boldsymbol{v}),$$

and the minimum is understood component-wise. Note the condition in (3.4) holds for any extreme-value copula with a continuously differentiable Pickands dependence function, see [Segers, 2012]. The following result describes the asymptotic properties of the new estimators $\widehat{A}_{n,h}$ for the Pickands dependence function. Weak convergence takes place in the space of all bounded functions on the unit simplex Δ_{d-1}, equipped with the topology induced by the sup-norm $\|\cdot\|_\infty$. The proof is given in the Appendix.

Theorem 3.1. *If the copula $C \geq \Pi$ satisfies condition (3.4), and the weight function h^* satisfies*

$$\left\| \frac{h^*}{\log} \right\|_\infty < \infty \quad \textit{and} \quad \int_0^1 h^*(y)\,(-\log y)^{-1}\,y^{-\lambda}\,dy < \infty \tag{3.6}$$

for some $\lambda > 1$, then we have for any $\gamma \in \left(\frac{1}{2}, \frac{\lambda}{2}\right)$ as $n \to \infty$

$$\sqrt{n}(\widehat{A}_{n,h} - A^*) \rightsquigarrow \mathbb{A}_{C,h} \tag{3.7}$$

in $\ell^\infty(\Delta_{d-1})$, where the limiting process is defined by

$$\mathbb{A}_{C,h} = B_h^{-1} \int_0^1 \frac{\mathbb{G}_C(y^{1-t_1-\ldots-t_{d-1}}, y^{t_1}, \ldots, y^{t_{d-1}})}{C(y^{1-t_1-\ldots-t_{d-1}}, y^{t_1}, \ldots, y^{t_{d-1}})} \frac{h^*(y)}{\log y}\,dy.$$

Remark 3.2. A careful inspection of the proof of this result shows that weak convergence of the empirical copula process lies at the heart of its proof. Since the latter converges under fairly more general conditions on the serial dependence of a stationary time series, the i.i.d. assumption on the series $\boldsymbol{X}_1, \ldots, \boldsymbol{X}_n$ can be easily dropped. Exploiting the results in [Doukhan et al., 2009] or [Bücher and Volgushev, 2013], the assertion of Theorem 3.1 holds true for every stationary sequence of random vectors provided Condition 2.1 in the latter reference is met. This condition is so mild that all usual concepts of weak serial dependence are included, e.g., strong mixing or absolute regularity of a time series at a mild polynomial decay of the corresponding coefficients. For details we refer to [Bücher and Volgushev, 2013]. The only difference to the i.i.d. case is reflected in a differing asymptotic covariance of the process $\mathbb{B}_C$ which is now given by

$$\mathrm{Cov}(\mathbb{B}_C(\boldsymbol{u}), \mathbb{B}_C(\boldsymbol{v})) = \sum_{j \in \mathbb{Z}} \mathrm{Cov}(\mathbb{I}\{\boldsymbol{U}_0 \leq \boldsymbol{u}\}, \mathbb{I}\{\boldsymbol{U}_j \leq \boldsymbol{v}\}),$$

and which, of course, reduces to $C(\boldsymbol{u} \wedge \boldsymbol{v}) - C(\boldsymbol{u})C(\boldsymbol{v})$ in the i.i.d. setting.

Note that the result of Theorem 3.1 is correct even in the case where C is not an extreme-value copula because the centering in (3.7) uses the best approximation with respect to the L^2-distance. The discussed estimator $\widehat{A}_{n,h}$ in general will neither be convex nor will it necessarily satisfy the boundary conditions of a multivariate Pickands dependence function. To ensure the latter restriction, one can replace the estimator $\hat{A}_{n,h}$ by the statistic

$$\max\left\{1 - \sum_{j=1}^{d-1} t_j, t_1, \ldots t_{d-1}, \min\{\hat{A}_{n,h}(\boldsymbol{t}), 1\}\right\}.$$

Furthermore, to provide convexity, the greatest convex minorant of this statistics can be used. As a consequence the estimator $\widehat{A}_{n,h}$ is replaced by a convex estimator with a smaller sup-norm between the true Pickands dependence function and the corresponding estimator, see [Marshall, 1970, Wang, 1986] and [Robertson et al., 1996]. An alternative way to achieve convexity and to correct for boundary properties of $\widehat{A}_{n,h}$ is the calculation of the L^2-projection on the space of partially linear functions satisfying these properties. This proposal was investigated by Fils-Villetard et al. [Fils-Villetard et al., 2008] and decreases the L^2-distance instead of the sup-norm.

Finally, we would like to point that none of these procedures guarantee that the modified estimator is in fact a Pickands dependence function. This is due to the fact that, for $d \geq 3$, a convex function satisfying the boundary conditions in (1.3) is not necessarily a Pickands dependence function. For an example see, e.g., [Beirlant et al., 2004].

4. A test for extreme-value dependence

Extreme-value copulas are often used as models for annual maximal data (see for instance [Yue, 2000] for the joint modeling of characteristic values of high floods, or [McNeil et al., 2005, Section 7.5.4] for financial applications) and it is advisable to check this model assumption in advance. Moreover, in order to reduce the number of candidate copula families that could be used as models for C in a general context, it is natural to test whether C belongs to the class of extreme-value copulas. To construct a test for this hypothesis we reconsider the L^2-distance $M_h(C,A^*)$ defined in (2.1). The following result will motivate the choice of the test statistic.

Lemma 4.1. *If h is a strictly positive weight function with $h^* \in L^1(0,1)$, then $C \geq \Pi^\kappa$ for some $\kappa \geq 1$ is an extreme-value copula if and only if*

$$\min\{M_h(C,A) \mid A \in \mathscr{A}\} = M_h(C,A^*) = 0.$$

Proof. If C is an extreme-value copula then A^* is the Pickands dependence function of the copula C and the weighted L^2-distance is equal to 0.

Now assume $M_h(C,A^*) = 0$. With the definition of the L^2-distance we obtain

$$\log C(y^{1-t_1-\dots-t_{d-1}}, y^{t_1}, \dots, y^{t_{d-1}}) = \log(y) A^*(\boldsymbol{t})$$

almost surely with respect to the Lebesgue measure on the set $(0,1) \times \Delta_{d-1}$. Since the functions $\log C(y^{1-t_1-\dots-t_{d-1}}, y^{t_1}, \dots, y^{t_{d-1}})$ and $(\log y) A^*(\boldsymbol{t})$ are continuous functions, the equality holds on the whole domain. This yields with a transformation for every $u_1, \dots, u_d \in (0,1]$

$$C(u_1,\dots,u_d) = \exp\Big\{\Big(\sum_{j=1}^d \log u_j\Big) A^*\Big(\frac{\log u_2}{\sum_{j=1}^d \log u_j}, \dots \frac{\log u_d}{\sum_{i=1}^d \log u_j}\Big)\Big\}.$$

and it can easily be shown that this identity also holds on the boundary. As a consequence, C is max-stable and thus an extreme-value copula. $\square$

Lemma 4.1 suggests to use $M_h(\tilde{C}_n, \hat{A}_{n,h})$ as a test statistic for the hypothesis

$$H_0: \ C \text{ is an extreme-value copula} \tag{4.1}$$

and to reject the null hypothesis for large values of $M_h(\tilde{C}_n, \hat{A}_{n,h})$. We now will investigate the asymptotic distribution of the test statistic under the null hypothesis and the alternative.

Theorem 4.2. *Let C be an extreme-value copula satisfying condition (3.4) with the Pickands dependence function A. If the weight function h is strictly positive, satisfies (3.6) and additionally the conditions*

$$\|h\|_\infty < \infty \quad and \quad \int_0^1 \frac{h(y)}{y^\lambda} dy < \infty \tag{4.2}$$

hold for some $\lambda > 2$, then we have for any $\gamma \in \left(\frac{1}{2}, \frac{\lambda}{4}\right)$ as $n \to \infty$

$$nM_h(\tilde{C}_n, \hat{A}_{n,h}) \rightsquigarrow Z_0,$$

where the non-negative random variable Z_0 is defined by

$$Z_0 := \int_{\Delta_{d-1}} \int_0^1 \left\{ \frac{\mathbb{G}_C(y^{1-t_1-\ldots-t_{d-1}}, y^{t_1}, \ldots, y^{t_{d-1}})}{C(y^{1-t_1-\ldots-t_{d-1}}, y^{t_1}, \ldots, y^{t_{d-1}})} \right\}^2 h(y)\, dy\, d\boldsymbol{t} - B_h \int_{\Delta_{d-1}} \mathbb{A}_{C,h}^2(\boldsymbol{t})\, d\boldsymbol{t},$$

and the constant B_h and the process $\mathbb{A}_{C,h}$ are defined in Theorem 3.1.

A plot of the density of Z_0 is given in Figure 1 for the Gumbel and the independence copula. The following result will give the asymptotic distribution of the test statistic under the alternative. Note that this is the case if and only if $M_h(C, A^*) > 0$.

Theorem 4.3. *Let $C \geq \Pi$ be a copula satisfying condition (3.4) such that $M_h(C, A^*) > 0$. If the strictly positive weight function h and the function h^* defined in (2.3) satisfy the conditions (3.6) and (4.2) for some $\lambda > 1$, then we have for any $\gamma \in (\frac{1}{2}, \frac{1+\lambda}{4} \wedge \frac{\lambda}{2})$ as $n \to \infty$*

$$\sqrt{n}(M_h(\tilde{C}_n, \hat{A}_{n,h}) - M_h(C, A^*)) \rightsquigarrow Z_1.$$

Here the random variable Z_1 is defined by

$$Z_1 := 2 \int_{\Delta_{d-1}} \int_0^1 \frac{\mathbb{G}_C(y^{1-t_1-\ldots-t_{d-1}}, y^{t_1}, \ldots, y^{t_{d-1}})}{C(y^{1-t_1-\ldots-t_{d-1}}, y^{t_1}, \ldots, y^{t_{d-1}})} \nu(y, \boldsymbol{t})\, dy\, d\boldsymbol{t}$$

with weight function

$$\nu(y, \boldsymbol{t}) = \left\{ \log C(y^{1-\sum_{i=1}^{d-1} t_i}, y^{t_1}, \ldots, y^{t_{d-1}}) - \log(y) A^*(\boldsymbol{t}) \right\} h(y).$$

Remark 4.4.
a) Again, the i.i.d. assumption on $\boldsymbol{X}_1, \ldots, \boldsymbol{X}_n$ in both preceding Theorems can be relaxed to weak serial dependence and strong stationarity, see Remark 3.2 in the previous section.
b) From Theorem 4.2 and 4.3 we obtain an asymptotic level α test for the hypothesis (4.1) by rejecting the null hypothesis H_0 if

$$nM_h(\tilde{C}_n, \hat{A}_{n,h}) > z_{1-\alpha},$$

where $z_{1-\alpha}$ denotes the $(1-\alpha)$-quantile of the distribution of the random variable Z_0. By Lemma 4.1 and Theorem 4.3 the test is (at least) consistent against all alternatives $C \geq \Pi$ satisfying assumption (3.4).
c) Approximating the integral in the definition of Z_1 in Theorem 4.3 by a Riemann sum, one can see that Z_1 is normally distributed with mean 0 and finite, positive variance, say σ^2. Consequently the power of the test is approximately given by

$$P(nM_h(\tilde{C}_n, \hat{A}_{n,h}) > z_{1-\alpha}) \quad \approx \quad 1 - \Phi\left(\frac{z_{1-\alpha}}{\sqrt{n}\sigma} - \sqrt{n}\frac{M_h(C, A^*)}{\sigma}\right) \approx \Phi\left(\sqrt{n}\frac{M_h(C, A^*)}{\sigma}\right),$$

where A^* is defined in (2.2) and Φ denotes the standard normal distribution function. Thus the power of the test is an increasing function with respect to n depending on the quantity $\frac{M_h(C, A^*)}{\sigma}$, see [Bücher et al., 2011].

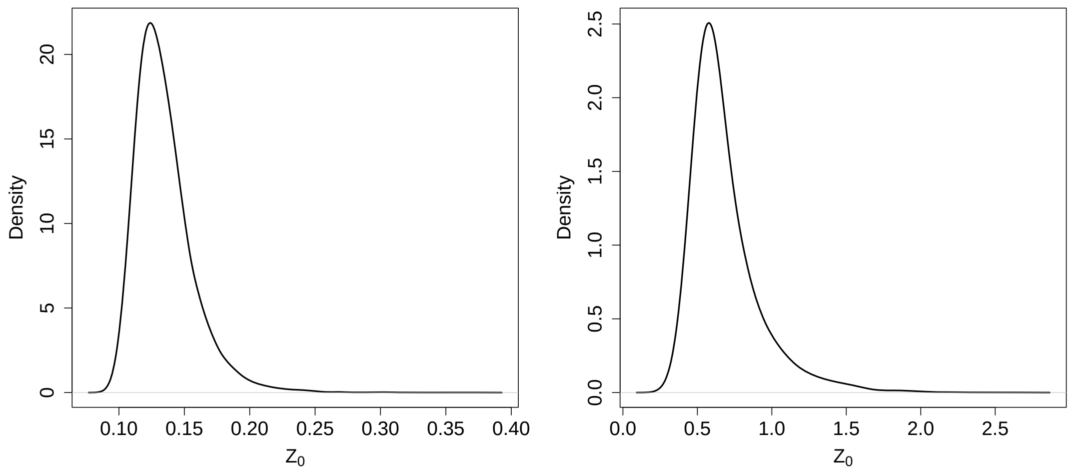

FIGURE 1. *Plot of the density of Z_0 for the Gumbel-copula with $\tau = 0.5$ (left) and the independence copula (right).*

For the construction of the test we need the $(1-\alpha)$-quantile of the distribution of the random variable Z_0. Unfortunately, this distribution depends on the unknown copula C and therefore it cannot be determined explicitly. However, the multiplier bootstrap initially proposed by Rémillard and Scaillet [Rémillard and Scaillet, 2009] and further investigated in [Bücher and Dette, 2010] and [Segers, 2012], provides a unified approach to obtain simulated approximate samples of the process $\mathbb{G}_C$. Plugging these into the defining integral of Z_0, we get approximate samples of Z_0 whose sample quantiles can finally serve as an approximation for the quantiles of Z_0.

More precisely, let $\widehat{\partial_j C_n}(\boldsymbol{u})$ be an estimator for $\partial_j C(\boldsymbol{u})$ which is uniformly bounded in n and $\boldsymbol{u}$ and which, for any $\delta \in (0, 1/2)$, satisfies the condition

$$\sup_{\boldsymbol{u}\in[0,1]^d : u_j \in[\delta,1-\delta]} |\widehat{\partial_j C_n}(\boldsymbol{u}) - \partial_j C(\boldsymbol{u})| \xrightarrow{\mathbb{P}} 0,$$

as $n \to \infty$, where $\xrightarrow{\mathbb{P}}$ denotes convergence in probability. It is easily seen, that for instance the following estimator based on finite differencing of the empirical copula satisfies these conditions:

$$\widehat{\partial_j C_n}(\boldsymbol{u}) = \begin{cases} \frac{C_n(\boldsymbol{u}+h_n\boldsymbol{e}_j)-C_n(\boldsymbol{u}-h_n\boldsymbol{e}_j)}{2h_n} & \text{if } u_j \in [h_n, 1-h_n] \\ \widehat{\partial_j C_n}(u_1,\dots,u_{j-1},h_n,u_{j+1},\dots,u_d) & \text{if } u_j \in [0,h_n) \\ \widehat{\partial_j C_n}(u_1,\dots,u_{j-1},1-h_n,u_{j+1},\dots,u_d) & \text{if } u_j \in (1-h_n,1], \end{cases}$$

where $h_n \to 0$ is a bandwidth such that $\inf_n h_n\sqrt{n} > 0$ and where $\boldsymbol{e}_j$ denotes the jth unit vector in $\mathbb{R}^d$.

Now, let $\xi_1, \xi_2, \ldots$ denote independent identically distributed random variables with mean 0 and variance 1 independent from $\boldsymbol{X}_1, \boldsymbol{X}_2, \ldots$ satisfying $\int_0^\infty \sqrt{P(|\xi_1| > x)}dx < \infty$. Define

$$\alpha_n^\xi(\boldsymbol{u}) = \frac{1}{\sqrt{n}} \sum_{i=1}^n \xi_i \left\{ \mathbb{I}\{\hat{U}_{i,1} \le u_1, \dots, \hat{U}_{i,d} \le u_d\} - C_n(\boldsymbol{u}) \right\}$$

and set

$$\mathbb{C}_n^{\xi}(\boldsymbol{u}) = \alpha_n^{\xi}(\boldsymbol{u}) - \sum_{j=1}^{d} \widehat{\partial_j C_n}(\boldsymbol{u})\, \alpha_n^{\xi}(1,\dots,1,u_j,1,\dots,1).$$

It follows from the results in [Segers, 2012] that if C satisfies condition (3.4), then

$$(\mathbb{C}_n, \mathbb{C}_n^{\xi}) \rightsquigarrow (\mathbb{G}_C, \mathbb{G}_C')$$

in $(\ell[0,1]^d, \|\cdot\|_\infty)^2$, where $\mathbb{G}_C$ denotes the process defined in (3.5) and $\mathbb{G}_C'$ is an independent copy of this process. By a similar reasoning as in [Bücher and Ruppert, 2013] we also obtain conditional weak convergence of $\mathbb{C}_n^{\xi}$ given the data in probability, which we denote by

$$\mathbb{C}_n^{\xi} \overset{\mathbb{P}}{\underset{\xi}{\rightsquigarrow}} \mathbb{G}_C.$$

For details on that type of convergence we refer to [Kosorok, 2008, Chapter 2.2.3]. Our final result now shows that the multiplier bootstrap procedure can be used to obtain a valid approximation for the distribution of the random variable Z_0.

Theorem 4.5. *Assume the copula $C \geq \Pi$ satisfies condition (3.4). If the weight function h satisfies the conditions in Theorem 4.2 and the function $y \mapsto h^*(y)(y\log y)^{-2}$ is uniformly bounded, then we get for the random variable*

$$\hat{Z}_n = \int_{\Delta_{d-1}} \int_0^1 \left\{ \frac{\mathbb{C}_n^{\xi}(y^{1-t_1-\dots-t_{d-1}}, y^{t_1}, \dots, y^{t_{d-1}})}{\tilde{C}_n(y^{1-t_1-\dots-t_{d-1}}, y^{t_1}, \dots, y^{t_{d-1}})} \right\}^2 h(y)\, dy\, d\boldsymbol{t}$$
$$- B_h^{-1} \int_{\Delta_{d-1}} \left\{ \int_0^1 \frac{\mathbb{C}_n^{\xi}(y^{1-t_1-\dots-t_{d-1}}, y^{t_1}, \dots, y^{t_{d-1}})}{\tilde{C}_n(y^{1-t_1-\dots-t_{d-1}}, y^{t_1}, \dots, y^{t_{d-1}})} \frac{h^*(y)}{\log y} dy \right\}^2 d\boldsymbol{t}$$

the weak conditional convergence $\hat{Z}_n \overset{\mathbb{P}}{\underset{\xi}{\rightsquigarrow}} Z_0$.

Proof. Due to the assumptions on the weight function all integrals in the definition of Z_0 are proper and therefore the mapping $(\mathbb{G}_C, C) \mapsto Z_0(\mathbb{G}_C, C)$ is continuous. Hence, the result follows from $\mathbb{C}_n^{\xi} \overset{\mathbb{P}}{\underset{\xi}{\rightsquigarrow}} \mathbb{G}_C$ and the continuous mapping theorem for the bootstrap, see, e.g., Theorem 10.8 in [Kosorok, 2008]. □

The bootstrap test is now obtained as follows. Repeating the procedure B times yields a sample $\hat{Z}_n(1), \dots, \hat{Z}_n(B)$ that is approximately distributed according to Z_0. This suggests to reject the null hypothesis if

$$nM_h(\tilde{C}_n, \hat{A}_{n,h}) > \hat{z}_{1-\alpha},$$

where $\hat{z}_{1-\alpha}$ denotes the empirical $(1-\alpha)$-quantile of this sample. It follows from Theorem 4.5 that the test holds its level α asymptotically and that it is consistent. The finite-sample performance of the test is investigated in the following section.

Remark 4.6. By the results in [Bücher and Ruppert, 2013] a block multiplier bootstrap can be used to obtain a valid bootstrap approximation of Z_0 in the case of strongly mixing stationary time series. We omit the details for the sake of brevity.

5. Finite-sample properties and illustration

This section is devoted to a simulation study regarding the finite-sample properties of the proposed estimators and tests for extreme-value copulas. We begin our discussion with the performance of the estimators. For that purpose we consider the trivariate extreme-value copula of logistic type as presented in [Tawn, 1990] with Pickands dependence function defined for $\boldsymbol{t} = (t_1, t_2) \in \Delta_2$ by

$$\begin{aligned} A(\boldsymbol{t}) = &(\theta^{1/\alpha} s_1^{1/\alpha} + \phi^{1/\alpha} s_2^{1/\alpha})^\alpha + (\theta^{1/\alpha} s_2^{1/\alpha} + \phi^{1/\alpha} s_3^{1/\alpha})^\alpha + (\theta^{1/\alpha} s_3^{1/\alpha} + \phi^{1/\alpha} s_1^{1/\alpha})^\alpha \\ &+ \psi(s_1^{1/\alpha} + s_2^{1/\alpha} + s_3^{1/\alpha})^\alpha + 1 - \theta - \phi - \psi, \end{aligned} \tag{5.1}$$

where $\boldsymbol{s} = (s_1, s_2, s_3) := (1 - t_1 - t_2, t_1, t_2)$ and $(\alpha, \theta, \phi, \psi) \in (0,1] \times [0,1]^3$. For the sake of comparison with existing simulation studies in the literature (see [Gudendorf and Segers, 2012]) we consider the parameters $(\theta, \phi, \psi) = (0,0,1)$ corresponding to a symmetric copula model (also widely known as the Gumbel–Hougaard copula) and $(\theta, \phi, \psi) = (0.6, 0.3, 0)$ corresponding to an asymmetric logistic copula. Furthermore, we also investigate the corresponding symmetric model in dimension 4. In this case the Pickands dependence function is defined by

$$A(\boldsymbol{t}) = (t_1^{1/\alpha} + t_2^{1/\alpha} + t_3^{1/\alpha} + (1 - t_1 - t_2 - t_3)^{1/\alpha})^\alpha$$

for $\boldsymbol{t} = (t_1, t_2, t_3) \in \Delta_3$. The parameter α was chosen from the set $\{0.3, 0.5, 0.7, 0.9\}$ which corresponds to bivariate marginal Kendall's tau in $\{0.7, 0.5, 0.3, 0.1\}$ and $\{0.21, 0.17, 0.11, 0.04\}$ for the symmetric and the asymmetric case, respectively. Note that in the model (5.1) all bivariate marginals have the same value of bivariate Kendall's τ.

We report Monte Carlo approximations for the mean integrated squared error (MISE)

$$\mathbb{E}\Big[\int_{\Delta_{d-1}} (\hat{A}(\boldsymbol{t}) - A(\boldsymbol{t}))^2 \mathrm{d}\boldsymbol{t}\Big],$$

where $\hat{A}$ represents successively the multivariate CFG-estimator, Pickands estimator (see [Gudendorf and Segers, 2012]) and the estimator introduced in the present paper which we abbreviate by BDV according to [Bücher et al., 2011]. The former two estimators are defined by the relationships

$$\frac{1}{A_n^{\mathrm{P}}(\boldsymbol{t})} = \frac{1}{n} \sum_{i=1}^n \hat{\xi}_i(\boldsymbol{t}), \qquad \text{and} \qquad \log A_n^{\mathrm{CFG}}(\boldsymbol{t}) = -\frac{1}{n} \sum_{i=1}^n \log \hat{\xi}_i(\boldsymbol{t}) - \gamma,$$

where

$$\hat{\xi}_i = \bigwedge_{j=1}^d \frac{-\log \hat{U}_{i,j}}{s_j} \qquad \text{with} \quad s_1 = 1 - \sum_{j=1}^{d-1} t_j \text{ and } s_j = t_{j-1} \text{ for } j \geq 2.$$

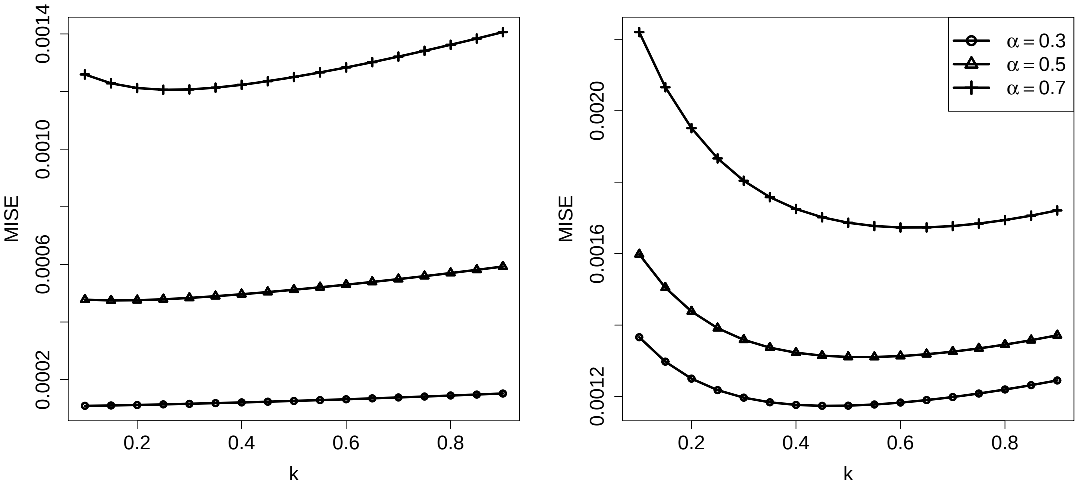

FIGURE 2. *Simulated MISE as a function of k for the symmetric (left) and the asymmetric (right) model for three different values of* α.

Regarding the choice of the weight function for the BDV-estimator we followed the proposal in [Bücher et al., 2011] and considered the function $h(y) = -y^k/\log(y)$ with $k = 0.5$. This choice seems to be a good compromise between a possibly difficult data-adaptive way of choosing a weight function and anaytical tractability; see Section 3.7 in [Bücher et al., 2011]. To confirm this choice, we simulated the mean integrated error for several choices of k. Exemplarily results for the symmetric and asymmetric model for dimension 3 are presented in Figure 2. Near $k = 0.5$ the MISE is minimized for the asymmetric model, whereas it stays relatively small for all values of k in the symmetric case. Furthermore, we see that the MISE increases for larger values of α, i.e. for smaller values of Kendall's τ. We also refer to Section 3.4 in [Bücher et al., 2011] for a discussion of "optimal" weight functions from an asymptotic point of view.
All estimators are corrected for the boundary conditions on the Pickands dependence function. Regarding the BDV-estimator we replace the initial estimate by the function

$$\max\left\{1-\sum_{j=1}^{d-1} t_j, t_1, \ldots t_{d-1}, \min\{\hat{A}_{n,h}(\boldsymbol{t}), 1\}\right\}.$$

For the Pickands- and CFG-estimator we used the endpoint-corrections proposed in [Gudendorf and Segers, 2012], who proposed a linear endpoint correction for $\log A(\boldsymbol{t})$. However, we did not correct the estimators with respect to their convexity. For each scenario we simulated 1.000 samples of size $n \in \{50, 100, 200\}$ using the simulation algorithms in [Stephenson, 2003] which are implemented in the R-package evd, [Stephenson, 2002]. To approximate the MISE, the estimators were evaluated on a grid of size 228 and 282 in dimension 3 and 4, respectively. The results are stated in Tables 1 - 3 and the main findings can be summarized as follows.

– The Pickands estimator is outperformed by the CFG and the BDV estimator. These findings are in accordance with the results of the simulation study in [Gudendorf and Segers, 2012].
– The CFG and the BDV estimator yield comparable results with slight advantages for the CFG estimator for strong dependence, whereas weak dependence results in more efficiency

for the BDV estimator. Since the BDV estimator is computationally more expensive, the CFG estimator might still be regarded as the "best" choice in practical applications.

TABLE 1. *Symmetric logistic dependence function in dimension 3,* $(\theta,\phi,\psi) = (0,0,1)$*: Simulated MISE for the Pickands, CFG- and BDV-estimator. The numbers in the brackts show the corresponding standard deviation.*

Sample size	Estimator	$\alpha = 0.3$	$\alpha = 0.5$	$\alpha = 0.7$	$\alpha = 0.9$
$n = 50$	P	2.37×10^{-4}	6.91×10^{-4}	1.70×10^{-3}	2.91×10^{-3}
		(9.9×10^{-6})	(1.1×10^{-4})	(5.2×10^{-5})	(6.6×10^{-5})
	CFG	9.94×10^{-5}	4.09×10^{-4}	1.16×10^{-3}	2.26×10^{-3}
		(4.6×10^{-6})	(1.4×10^{-5})	(3.5×10^{-5})	(6.6×10^{-5})
	BDV	1.24×10^{-4}	5.07×10^{-4}	1.27×10^{-3}	2.04×10^{-3}
		(4.9×10^{-6})	(1.7×10^{-5})	(3.8×10^{-5})	(5.5×10^{-5})
$n = 100$	P	1.01×10^{-4}	3.31×10^{-4}	7.59×10^{-4}	1.43×10^{-3}
		(3.7×10^{-6})	(1.3×10^{-5})	(2.2×10^{-5})	(3.1×10^{-5})
	CFG	4.12×10^{-5}	2.28×10^{-4}	6.04×10^{-4}	1.17×10^{-3}
		(1.4×10^{-6})	(8.2×10^{-6})	(2.0×10^{-5})	(3.7×10^{-5})
	BDV	5.46×10^{-5}	2.69×10^{-4}	6.23×10^{-4}	1.01×10^{-3}
		(1.7×10^{-6})	(9.1×10^{-6})	(2.0×10^{-5})	(2.9×10^{-5})
$n = 200$	P	4.69×10^{-5}	1.59×10^{-4}	3.92×10^{-4}	7.15×10^{-4}
		(1.6×10^{-6})	(4.9×10^{-6})	(1.1×10^{-5})	(1.6×10^{-5})
	CFG	2.34×10^{-5}	1.07×10^{-4}	3.02×10^{-4}	5.21×10^{-4}
		(7.7×10^{-7})	(3.9×10^{-6})	(9.8×10^{-6})	(1.7×10^{-5})
	BDV	2.84×10^{-5}	1.20×10^{-4}	2.93×10^{-4}	4.77×10^{-4}
		(8.7×10^{-7})	(4.1×10^{-6})	(9.0×10^{-6})	(1.3×10^{-5})

TABLE 2. *Asymmetric logistic dependence function in dimension 3,* $(\theta,\phi,\psi) = (0.6,0.3,0)$*: Simulated MISE for the Pickands, CFG- and BDV-estimator. The numbers in the brackts show the corresponding standard deviation.*

Sample size	Estimator	$\alpha = 0.3$	$\alpha = 0.5$	$\alpha = 0.7$	$\alpha = 0.9$
$n = 50$	P	1.65×10^{-3}	1.98×10^{-3}	2.49×10^{-3}	3.13×10^{-3}
		(4.4×10^{-5})	(4.9×10^{-5})	(5.5×10^{-5})	(6.2×10^{-5})
	CFG	1.10×10^{-3}	1.32×10^{-3}	1.77×10^{-3}	2.51×10^{-3}
		(2.4×10^{-5})	(3.0×10^{-54})	(4.6×10^{-5})	(6.4×10^{-5})
	BDV	1.19×10^{-3}	1.34×10^{-3}	1.67×10^{-3}	2.16×10^{-3}
		(2.6×10^{-5})	(3.0×10^{-5})	(4.2×10^{-5})	(5.9×10^{-5})
$n = 100$	P	8.55×10^{-4}	9.48×10^{-4}	1.23×10^{-3}	1.53×10^{-3}
		(2.1×10^{-5})	(2.2×10^{-5})	(2.7×10^{-5})	(3.3×10^{-5})
	CFG	5.42×10^{-4}	6.56×10^{-4}	8.32×10^{-4}	1.19×10^{-3}
		(1.2×10^{-5})	(1.6×10^{-5})	(2.2×10^{-5})	(3.4×10^{-5})
	BDV	5.69×10^{-4}	6.61×10^{-4}	8.04×10^{-4}	9.86×10^{-4}
		(1.3×10^{-5})	(1.6×10^{-5})	(2.0×10^{-5})	(2.9×10^{-5})
$n = 200$	P	4.05×10^{-4}	4.59×10^{-4}	5.99×10^{-4}	7.45×10^{-4}
		(9.0×10^{-6})	(1.1×10^{-5})	(1.2×10^{-5})	(1.6×10^{-5})
	CFG	2.85×10^{-4}	3.20×10^{-4}	4.13×10^{-4}	5.28×10^{-4}
		(6.3×10^{-6})	(7.9×10^{-6})	(1.1×10^{-5})	(1.6×10^{-5})
	BDV	2.91×10^{-4}	3.34×10^{-4}	3.90×10^{-4}	4.67×10^{-4}
		(6.5×10^{-6})	(8.3×10^{-6})	(9.9×10^{-6})	(1.2×10^{-5})

Finally, we conducted Monte Carlo experiments to investigate the level and the power of the test for extreme-value dependence introduced in Section 4. We fixed the dimension to $d = 3$ and

TABLE 3. *Symmetric logistic dependence function in dimension 4: Simulated MISE for the Pickands, CFG- and BDV-estimator. The numbers in the brackts show the corresponding standard deviation.*

Sample size	Estimator	$\alpha = 0.3$	$\alpha = 0.5$	$\alpha = 0.7$	$\alpha = 0.9$
$n = 50$	P	1.34×10^{-4}	2.52×10^{-4}	5.37×10^{-4}	9.19×10^{-4}
		(3.1×10^{-6})	(8.8×10^{-6})	(1.6×10^{-5})	(2.2×10^{-5})
	CFG	3.16×10^{-5}	1.60×10^{-4}	4.72×10^{-4}	9.29×10^{-4}
		(1.1×10^{-6})	(5.6×10^{-6})	(1.3×10^{-5})	(2.5×10^{-5})
	BDV	4.49×10^{-5}	2.05×10^{-4}	5.15×10^{-4}	7.78×10^{-4}
		(1.2×10^{-6})	(7.1×10^{-6})	(1.5×10^{-5})	(2.0×10^{-5})
$n = 100$	P	4.88×10^{-5}	1.19×10^{-4}	2.61×10^{-4}	4.76×10^{-4}
		(1.2×10^{-6})	(4.1×10^{-6})	(6.9×10^{-6})	(1.1×10^{-5})
	CFG	1.67×10^{-5}	8.65×10^{-5}	2.36×10^{-4}	4.52×10^{-4}
		(6.3×10^{-7})	(3.0×10^{-6})	(7.4×10^{-6})	(1.4×10^{-5})
	BDV	2.35×10^{-5}	1.05×10^{-4}	2.41×10^{-4}	3.85×10^{-4}
		(8.0×10^{-7})	(3.6×10^{-6})	(7.4×10^{-6})	(1.0×10^{-5})
$n = 200$	P	2.01×10^{-6}	5.27×10^{-5}	1.35×10^{-4}	2.41×10^{-4}
		(5.4×10^{-7})	(1.3×10^{-6})	(3.6×10^{-6})	(5.5×10^{-6})
	CFG	7.88×10^{-6}	4.40×10^{-5}	1.26×10^{-4}	2.22×10^{-4}
		(3.1×10^{-7})	(1.4×10^{-6})	(4.1×10^{-6})	(7.1×10^{-6})
	BDV	1.04×10^{-5}	5.10×10^{-5}	1.25×10^{-4}	1.98×10^{-4}
		(3.6×10^{-7})	(1.5×10^{-6})	(4.0×10^{-6})	(5.4×10^{-6})

considered samples of size $n = 200$ and $n = 400$ where the level of the test is 5%. Under the null hypothesis, we simulated data from the symmetric and asymmetric logistic type model as defined in (5.1) with parameters $(\theta, \phi, \psi) = (0,0,1)$ (i.e., the Gumbel–Hougaard copula) and $(\theta, \phi, \psi) = (0.6, 0.3, 0)$, respectively. For the sake of an easy comparison with the two-dimensional version of the test in [Bücher et al., 2011] and with the extensive simulation study in [Kojadinovic et al., 2011] we chose the remaining parameter α in the Gumbel–Hougaard model in such a way that Kendall's τ of all bivariate marginal varies in the set $\{0.25, 0.5, 0.75\}$. In the asymmetric case, the supremum of all achievable values of τ is 0.25, whence we consider $\tau \in \{0.05, 0.15, 0.249\}$ in this case. Under the alternative we considered the Clayton, Frank, Normal and t-copula with four degrees of freedom and Kendall's $\tau \in \{0.25, 0.5, 0.75\}$. In order to get an impression of how "far" these models are in the alternative, we provide the minimal distance $M_h(C, A^*)$ from the model under investigation to its best approximation in the class of extreme-value copulas in Table 4. The estimator in the test statistic $nM_h(\tilde{C}_n, \hat{A}_{n,h})$ has not been corrected for the boundary conditions as described in the previous paragraph, since this could have an unknown influence on its asymptotic distribution. For the multiplier method we chose $B = 250$ Bootstrap-replicates and we used the bandwidth $h_n = 1/\sqrt{n}$ for the estimators of the partial derivatives. The test was carried out at the 5% significance level and empirical rejection rates were computed from 1.000 random samples in each scenario. The results are stated in Table 4 and the main findings are as follows.

- The test seems to be globally conservative, although the observed level improves with increasing sample size. This effect is observed to be stronger for increasing level of dependence (measured by Kendall's τ) and is in accordance with other simulation studies on the multiplier method for copulas with strong dependence.
- In terms of power the test detects all alternatives with reasonable rejection rates. As expected, the power increases with a larger distance $M_h(C, A^*)$. Similar observations have been made

TABLE 4. *Simulated rejection probabilities of the test for the null hypothesis of an extreme-value copula where the level is 5%.*

Copula	τ	$M_h(C,A^*)$	$n=200$	$n=400$
Gumbel	0.25	0	0.022	0.047
	0.5	0	0.005	0.014
	0.75	0	0	0
Asym. log.	0.05	0	0.029	0.034
	0.15	0	0.017	0.023
	0.249	0	0.009	0.012
Clayton	0.25	$1.99\cdot10^{-3}$	0.934	1
	0.5	$3.06\cdot10^{-3}$	1	1
	0.75	$1.83\cdot10^{-3}$	0.997	1
Frank	0.25	$8.19\cdot10^{-4}$	0.426	0.949
	0.5	$1.32\cdot10^{-3}$	0.853	1
	0.75	$7.46\cdot10^{-4}$	0.686	1
Normal	0.25	$4.89\cdot10^{-4}$	0.192	0.677
	0.50	$3.72\cdot10^{-4}$	0.184	0.721
	0.75	$6.78\cdot10^{-5}$	0.006	0.068
t-Copula	0.25	$1.51\cdot10^{-4}$	0.037	0.162
	0.5	$1.64\cdot10^{-4}$	0.047	0.297
	0.75	$3.62\cdot10^{-5}$	0.001	0.012

in the two-dimensional case in [Bücher et al., 2011].

Finally, note that the test is computationally quite extensive which makes it hard to be implemented in higher dimensions. This fact hints at a possible future research project since a more feasible test for higher dimensions is clearly needed.

As an illustration, we applied the proposed test on the uranium exploration data of Cook and Johnson [Cook and Johnson, 1986], which consists of log concentrations of seven chemical elements (Uranium (U), Lithium (Li), Cobalt (Co), Potassium (K), Cesium (Cs), Scandium (Sc), Titanium (Ti)) in 655 water samples collected near Grand Junction, Colorado. For the sake of comparison with the results in [Kojadinovic et al., 2011], we tested for three-dimensional extreme-value dependence of the triples {U,Co,Li}, {U,Li,Ti} and {Ti,Li,Cs}. To deal with the problem of a non-negligible number of ties in this data set, we followed the proposal in [Kojadinovic and Yan, 2010a] and assigned the ranks at random using the R function `rank` with its argument `ties.method=random`. We repeated this randomized procedure 100 times with 250 bootstrap replications and state the minimum, median and maximum of the obtained p-values in Table 5. Furthermore, we carried out the test using mid-ranks with 2000 bootstrap replications. The results in Table 5 show strong evidence against extreme-value dependence for two of the triples. For the tripel {Ti,Li,Cs}, there is only minor evidence against extreme value dependence. Overall, this is in accordance with the findings in [Kojadinovic et al., 2011].

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TABLE 5. *Approximate p-values for the test for three-dimensional extreme-value dependence obtained for the triples {U,Co,Li}, {U,Li,Ti} and {Ti,Li,C} of the uranium data set in [Cook and Johnson, 1986]*

	Random ranks for ties			
	minimum	median	maximum	Mid-ranks
{ U,Co,Li }	0	0.005	0.02	0.0015
{ U,Li,Ti }	0	0.005	0.02	0.0005
{ Ti,Li,Cs }	0.02	0.055	0.11	0.02

6. Proofs

6.1. Proof of Theorem 3.1

The proof follows from a slightly more general result, which establishes weak convergence for the weighted process

$$W_{n,\omega}(\boldsymbol{t}) = \int_0^1 \log \frac{\tilde{C}_n(y^{1-t_1-\ldots-t_{d-1}}, y^{t_1}, \ldots, y^{t_{d-1}})}{C(y^{1-t_1-\ldots-t_{d-1}}, y^{t_1}, \ldots, y^{t_{d-1}})}\, \omega(y,\boldsymbol{t})\, dy,$$

where the weight function $\omega : [0,1] \times \Delta_{d-1}$ may depend on y and $\boldsymbol{t}$. Theorem 3.1 is a simple consequence of the following result using the weight function $\omega(y,\boldsymbol{t}) = B_h^{-1} \frac{h^*(y)}{\log y}$.

Theorem 6.1. *Assume that for the weight function $\omega : [0,1] \times \Delta_{d-1} \to \overline{\mathbb{R}}$ there exists a bounded function $\overline{\omega} : [0,1] \to \mathbb{R}_0^+$ such that $|\omega(y,\boldsymbol{t})| \leq \overline{\omega}(y)$ for all $y \in [0,1]$ and all $\boldsymbol{t} \in \Delta_{d-1}$ and such that*

$$\int_0^1 \overline{\omega}(y) y^{-\lambda} dy < \infty \text{ for some } \lambda > 1. \tag{6.1}$$

If the copula $C \geq \Pi$ satisfies (3.4) then we have for every $\gamma \in \left(\frac{1}{2}, \frac{\lambda}{2}\right)$ as $n \to \infty$

$$\sqrt{n} W_{n,\omega}(\boldsymbol{t}) \rightsquigarrow \mathbb{W}_{C,\omega}(\boldsymbol{t}) \quad \text{in } \ell^\infty(\Delta_{d-1}),$$

where the limiting process is given by

$$\mathbb{W}_{C,\omega}(\boldsymbol{t}) = \int_0^1 \frac{\mathbb{G}_C(y^{1-t_1-\ldots-t_{d-1}}, y^{t_1}, \ldots, y^{t_{d-1}})}{C(y^{1-t_1-\ldots-t_{d-1}}, y^{t_1}, \ldots, y^{t_{d-1}})}\, \omega(y,\boldsymbol{t})\, dy.$$

Proof of Theorem 6.1. Fix $\lambda > 1$ and $\gamma \in \left(\frac{1}{2}, \frac{\lambda}{2}\right)$. Due to Lemma 1.10.2 in [van der Vaart and Wellner, 1996], the processes $\sqrt{n}(\tilde{C}_n - C)$ and $\sqrt{n}(C_n - C)$ will have the same weak limit. For $i = 1, 2, \ldots$ we consider the following random functions in $\ell^\infty(\Delta_{d-1})$:

$$W_n(\boldsymbol{t}) := \int_0^1 \sqrt{n}\big\{ \log \tilde{C}_n(y^{1-t_1-\ldots-t_{d-1}}, y^{t_1}, \ldots, y^{t_{d-1}}) \\ - \log C(y^{1-t_1-\ldots-t_{d-1}}, y^{t_1}, \ldots, y^{t_{d-1}})\big\}\, \omega(y,\boldsymbol{t})\, dy$$

$$W_{i,n}(\boldsymbol{t}) := \int_{1/i}^1 \sqrt{n}\big\{ \log \tilde{C}_n(y^{1-t_1-\ldots-t_{d-1}}, y^{t_1}, \ldots, y^{t_{d-1}})$$

$$-\log C(y^{1-t_1-\ldots-t_{d-1}}, y^{t_1}, \ldots, y^{t_{d-1}})\big\}\,\omega(y,\boldsymbol{t})\,dy$$

$$W(\boldsymbol{t}) := \int_0^1 \frac{\mathbb{G}_C(y^{1-t_1-\ldots-t_{d-1}}, y^{t_1}, \ldots, y^{t_{d-1}})}{C(y^{1-t_1-\ldots-t_{d-1}}, y^{t_1}, \ldots, y^{t_{d-1}})}\,\omega(y,\boldsymbol{t})\,dy$$

$$W_i(\boldsymbol{t}) := \int_{1/i}^1 \frac{\mathbb{G}_C(y^{1-t_1-\ldots-t_{d-1}}, y^{t_1}, \ldots, y^{t_{d-1}})}{C(y^{1-t_1-\ldots-t_{d-1}}, y^{t_1}, \ldots, y^{t_{d-1}})}\,\omega(y,\boldsymbol{t})\,dy$$

With this notation we have to show the following three assertions :

(i) $W_{i,n} \rightsquigarrow W_i$ in $\ell^\infty(\Delta_{d-1})$ for $n \to \infty$,

(ii) $W_i \rightsquigarrow W$ in $\ell^\infty(\Delta_{d-1})$ for $i \to \infty$,

(iii) for every $\varepsilon > 0$: $\lim_{i\to\infty} \overline{\lim}_{n\to\infty} \mathbb{P}^* \left(\sup_{t\in\Delta_{d-1}} |W_{i,n}(t) - W_n(t)| > \varepsilon \right) = 0$,

then Lemma B.1 in [Bücher et al., 2011] yields the convergence $W_n \rightsquigarrow W$ in $\ell^\infty(\Delta_{d-1})$.

We begin with the proof of assertion (i). For this purpose we set $T_i = [1/i, 1]^d$ for $i \in \mathbb{N}$ and consider the mapping

$$\Phi_1 : \begin{cases} \mathbb{D}_{\Phi_1} \to \ell^\infty(T_i) \\ f \mapsto \log \circ f, \end{cases}$$

where the domain is defined by $\mathbb{D}_{\Phi_1} := \{f \in \ell^\infty(T_i) \mid \inf_{\boldsymbol{x}\in T_i} |f(\boldsymbol{x})| > 0\}$. Due to Lemma 12.2 in [Kosorok, 2008], it follows that Φ_1 is Hadamard-differentiable at C tangentially to $\ell^\infty(T_i)$ with derivative $\Phi'_{1,C}(f) = \frac{f}{C}$. Since $\tilde{C}_n \geq n^{-\gamma}$ and $C \geq \Pi$, we have $\tilde{C}_n, C \in \mathbb{D}_{\Phi_1}$ and with the functional delta method we obtain

$$\sqrt{n}(\log \tilde{C}_n - \log C) \rightsquigarrow \frac{\mathbb{G}_C}{C}$$

for $n \to \infty$ in $\ell^\infty(T_i)$. Now we consider the mapping

$$\Phi_2 : \begin{cases} \ell^\infty(T_i) \to \ell^\infty([1/i, 1] \times \Delta_{d-1}) \\ f \mapsto f \circ \phi \end{cases} ,$$

where the mapping $\phi : [1/i, 1] \times \Delta_{d-1} \to T_i$ is defined by

$$\phi(y, \boldsymbol{t}) = (y^{1-t_1-\ldots-t_{d-1}}, y^{t_1}, \ldots, y^{t_{d-1}}).$$

For Φ_2 the following inequality holds:

$$\begin{aligned} \|\Phi_2(f) - \Phi_2(g)\|_\infty &= \sup_{y\in[1/i,1], \boldsymbol{t}\in\Delta_{d-1}} |f \circ \phi(y, \boldsymbol{t}) - g \circ \phi(y, \boldsymbol{t})| \\ &\leq \sup_{\boldsymbol{x}\in T_i} |f(\boldsymbol{x}) - g(\boldsymbol{x})| = \|f - g\|_\infty. \end{aligned}$$

This implies that Φ_2 is Lipschitz-continuous. By the continuous mapping theorem and the boundedness of the weight function ω we obtain

$$\sqrt{n}\left\{\log \tilde{C}_n(y^{1-t_1-\ldots-t_{d-1}}, y^{t_1}, \ldots, y^{t_{d-1}}) - \log C(y^{1-t_1-\ldots-t_{d-1}}, y^{t_1}, \ldots, y^{t_{d-1}})\right\} \omega(y, \boldsymbol{t})$$

$$\rightsquigarrow \frac{\mathbb{G}_C(y^{1-t_1-\ldots-t_{d-1}}, y^{t_1}, \ldots, y^{t_{d-1}})}{C(y^{1-t_1-\ldots-t_{d-1}}, y^{t_1}, \ldots, y^{t_{d-1}})} \omega(y, \boldsymbol{t})$$

in $\ell^\infty\left([\frac{1}{i}, 1] \times \Delta_{d-1}\right)$. By integration with respect to $y \in [1/i, 1]$ assertion (i) follows.

Assertion (ii) follows directly from the observation, that the process $\mathbb{G}_C$ is bounded on $[0,1]^d$ and from the fact, that the function

$$\boldsymbol{t} \mapsto \frac{\omega(y, \boldsymbol{t})}{C(y^{1-t_1-\ldots-t_{d-1}}, y^{t_1}, \ldots, y^{t_{d-1}})}$$

can be bounded by the integrable function $\overline{\omega}(y)\, y^{-1}$. The proof of (iii) is obtained by the same arguments as given in [Bücher et al., 2011] in the case $d = 2$ and is therefore omitted. □

6.2. *Proof of Theorem 4.2*

Since integration is continuous, it suffices to show the weak convergence $\bar{W}_n(t) \rightsquigarrow \bar{W}(t)$ in $\ell^\infty(\Delta_{d-1})$, where we define

$$\bar{W}_n(\boldsymbol{t}) = \int_0^1 n\left(\log \frac{\tilde{C}_n(y^{1-t_1-\ldots-t_{d-1}}, y^{t_1}, \ldots, y^{t_{d-1}})}{C(y^{1-t_1-\ldots-t_{d-1}}, y^{t_1}, \ldots, y^{t_{d-1}})}\right)^2 h(y)\, dy - nB_h(\hat{A}_{n,h}(\boldsymbol{t}) - A(\boldsymbol{t}))^2$$

$$\bar{W}(\boldsymbol{t}) = \int_0^1 \left(\frac{\mathbb{G}_C(y^{1-t_1-\ldots-t_{d-1}}, y^{t_1}, \ldots, y^{t_{d-1}})}{C(y^{1-t_1-\ldots-t_{d-1}}, y^{t_1}, \ldots, y^{t_{d-1}})}\right)^2 h(y)\, dy - B_h \mathbb{A}_{C,h}^2(\boldsymbol{t}).$$

Now we will proceed similar to the proof of Theorem 6.1 and consider

$$\begin{aligned}
\bar{W}_{i,n}(\boldsymbol{t}) = &\int_{1/i}^1 n\left(\log \frac{\tilde{C}_n(y^{1-t_1-\ldots-t_{d-1}}, y^{t_1}, \ldots, y^{t_{d-1}})}{C(y^{1-t_1-\ldots-t_{d-1}}, y^{t_1}, \ldots, y^{t_{d-1}})}\right)^2 h(y)\, dy \\
&- B_h^{-1}\left(\int_{1/i}^1 \sqrt{n} \log \frac{\tilde{C}_n(y^{1-t_1-\ldots-t_{d-1}}, y^{t_1}, \ldots, y^{t_{d-1}})}{C(y^{1-t_1-\ldots-t_{d-1}}, y^{t_1}, \ldots, y^{t_{d-1}})} \frac{h^*(y)}{\log y}\, dy\right)^2 \\
\bar{W}_i(\boldsymbol{t}) = &\int_{1/i}^1 \left(\frac{\mathbb{G}_C(y^{1-t_1-\ldots-t_{d-1}}, y^{t_1}, \ldots, y^{t_{d-1}})}{C(y^{1-t_1-\ldots-t_{d-1}}, y^{t_1}, \ldots, y^{t_{d-1}})}\right)^2 h(y)\, dy \\
&- B_h^{-1}\left(\int_{1/i}^1 \frac{\mathbb{G}_C(y^{1-t_1-\ldots-t_{d-1}}, y^{t_1}, \ldots, y^{t_{d-1}})}{C(y^{1-t_1-\ldots-t_{d-1}}, y^{t_1}, \ldots, y^{t_{d-1}})} \frac{h^*(y)}{\log y}\, dy\right)^2.
\end{aligned}$$

Due to Lemma B.1 in [Bücher et al., 2011] it suffices to show

(i) $\bar{W}_{i,n} \rightsquigarrow \bar{W}_i$ in $\ell^\infty(\Delta_{d-1})$ for $n \to \infty$,

(ii) $\bar{W}_i \rightsquigarrow \bar{W}$ in $\ell^\infty(\Delta_{d-1})$ for $i \to \infty$,

(iii) for every $\varepsilon > 0$: $\lim_{i\to\infty} \overline{\lim}_{n\to\infty} \mathbb{P}^*\left(\sup_{t\in\Delta_{d-1}} |\bar{W}_{i,n}(t) - \bar{W}_n(t)| > \varepsilon\right) = 0$.

The proof of these assertions follows by similar arguments as in [Bücher et al., 2011] and is omitted for the sake of brevity. □

6.3. Proof of Theorem 4.3

We use the decomposition

$$M_h(\tilde{C}_n, \hat{A}_{n,h}) - M_h(C, A^*) = S_1 + S_2 + S_3, \tag{6.2}$$

where

$$S_1 = 2\int_{\Delta_{d-1}} \int_0^1 \left\{\bar{C}_n(y,\boldsymbol{t}) - \bar{C}(y,\boldsymbol{t})\right\}\left\{\bar{C}(y,\boldsymbol{t}) - A^*(\boldsymbol{t})(-\log y)\right\} h(y)\, dy\, d\boldsymbol{t},$$
$$S_2 = \int_{\Delta_{d-1}} \int_0^1 \left\{\bar{C}_n(y,\boldsymbol{t}) - \bar{C}(y,\boldsymbol{t})\right\}^2 h(y)\, dy\, d\boldsymbol{t}$$
$$S_3 = -B_h \int_{\Delta_{d-1}} \left\{\hat{A}_{n,h}(\boldsymbol{t}) - A^*(\boldsymbol{t})\right\}^2 d\boldsymbol{t}$$

and we used the notations

$$\bar{C}(y,\boldsymbol{t}) = -\log C(y^{1-t_1-\ldots-t_{d-1}}, y^{t_1}, \ldots, y^{t_{d-1}}),$$
$$\bar{C}_n(y,\boldsymbol{t}) = -\log \tilde{C}_n(y^{1-t_1-\ldots-t_{d-1}}, y^{t_1}, \ldots, y^{t_{d-1}}).$$

To investigate the convergence of the first term in (6.2) we first notice that $|\nu(y,\boldsymbol{t})| \leq \bar{\nu}(y)$, with $\bar{\nu}(y) := \frac{2h^*(y)}{-\log y}$. The assumptions of the theorem on the weight function h imply that we van invoke Theorem 6.1. With the continuous mapping theorem this yields $\sqrt{n}S_1 \rightsquigarrow Z_1$ and it remains to show that the remaining two terms S_2 and S_3 can be neglected. By Theorem 3.1 and the continuous mapping theorem we have $S_3 = O_P\left(\frac{1}{n}\right)$ and finally S_2 can be estimated along similar lines as in the proof of Theorem 4.2 in [Bücher et al., 2011]. □

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