Journal de la Société Française de Statistique
Vol. 154 No. 1 (2013)

Practical Notes On Multivariate Modeling Based on Elliptical Copulas

Titre: Aspects pratiques de la modélisation multivariée fondée sur les copules elliptiques

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Abstract: Multivariate distributions based on elliptical copulas have been widely used in many fields such as hydrology and finance. We focus on two practical issues of applications of such models. The first is a caveat rooted in a consistency property defined by Kano (1994, Journal of Multivariate Analysis, 51:139–147) for elliptical distributions. Some elliptical families do not have this property, which puts practical limitations on applications and software implementation of the corresponding elliptical copulas. The second issue is on conditional sampling from such distributions, which is important in Monte Carlo statistical inferences, especially when closed-form solutions are not available or feasible. Two sampling methods are presented: a direct sampling approach based on a stochastic representation of elliptical distributions, and an acceptance/rejection sampling method. The latter also provides an importance sampler as a byproduct, which may have higher efficiency for some applications. A trivariate model of the volume, duration, and peak intensity of annual extreme storms illustrates the sampling algorithms.

Résumé : Les distributions multivariées construites à partir de copules elliptiques sont utilisées dans de nombreux domaines comme l'hydrologie et la finance. Nous nous intéressons à deux aspects pratiques concernant ces modèles. Dans un premier temps, nous attirons l'attention sur l'importance de la propriété de consistance définie par Kano (1994, Journal of Multivariate Analysis, 51 :139–147). Certaines distributions elliptiques ne satisfont pas cette propriété, ce qui limite les applications et l'implantation des copules correspondantes dans les logiciels. Dans un deuxième temps, nous donnons deux méthodes conditionnelles pour la génération d'échantillons aléatoires à partir de distributions elliptiques. La première approche présentée est fondée sur une représentation stochastique des distributions elliptiques alors que la seconde utilise une méthode d'acceptation/rejet. L'utilisation des deux méthodes est illustrée dans la cadre de la modélisation de données hydrologiques trivariées.

Keywords: conditional sampling, elliptical distribution, importance sampling, marginal consistency
Mots-clés : distribution elliptique, génération conditionnelle d'échantillons aléatoires, propriété de consistance
AMS 2000 subject classifications: 46N30, 65C10

1. Introduction

Copula-based multivariate modeling has found extensive applications in many fields such as finance (e.g., Cherubini et al., 2004; McNeil et al., 2005), insurance (e.g., Frees and Valdez, 1998), and hydrology (e.g., Genest and Favre, 2007; Genest et al., 2007). By Sklar's theorem, the cumulative distribution function (CDF) F of any continuous p-dimensional random vector $\mathbf{Y} = (Y_1, \ldots, Y_p)^\top$ has a unique representation

$$F(y_1, \ldots, y_p) = C\{F_1(y_1), \ldots, F_p(y_p)\} \tag{1}$$

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where F_i is the CDF of Y_i, $i = 1, \dots, p$, and $C : [0,1]^p \to [0,1]$, called a copula, is a p-dimensional CDF with standard uniform margins (Sklar, 1959). This representation suggests a natural two-part multivariate model: a collection of marginal distributions and an accompanying copula. Detailed reviews of copulas can be found, for example, in Joe (1997) and Nelsen (2006).

Elliptical copulas are a class of copulas of great practical importance. In comparison with another class known as Archimedean copulas (e.g., Genest and MacKay, 1986), the elliptical class offers a good compromise between convenience and flexibility (e.g., Genest et al., 2007). An elliptical copula is the implicit copula that is uniquely determined by an elliptical distribution. A multivariate distribution for a random vector $\mathbf{Y}$ with given marginal distributions $F_1, \dots, F_p$ can be constructed with (1) using an elliptical copula C. Such a multivariate distribution of $\mathbf{Y}$ is called meta-elliptical distribution (Fang et al., 2002; Abdous et al., 2005).

We focus on two practical issues on meta-elliptical distributions. The first is a caveat rooted in the marginal consistency property for elliptical distributions studied by Kano (1994). This property is that the marginal distributions of an elliptical distribution belong to the same elliptical family; see formal definition in equation (7) in Section 3. Nonetheless, some elliptical families do not have this property. Consider the elliptical exponential power distribution as an example. The bivariate marginal distribution of the first two components of a p-dimensional exponential power distribution is not itself a bivariate exponential power distribution. Furthermore, the bivariate distribution changes with p. This is in contrast to the case of p-dimensional normal distribution whose marginal distributions are always normal regardless of p. An elliptical family without the marginal consistency property has limitations in applications and software implementation of the corresponding meta-elliptical distribution.

The second issue is on conditional sampling from such distributions, which is important in Monte Carlo statistical inferences such as risk analysis, power study and parametric bootstrap, especially when closed-form solutions are not available or feasible. For example, consider a trivariate hydrological application. Structural design engineers demand detailed knowledge of three major storm characteristics: peak intensity, volume, and duration. Given volume and duration at certain levels, the conditional distribution of peak intensity is of great hydrological interest. A large sample from the conditional distribution can be used to assess risks in designing flood protection structures.

This article is organized as follows. Elliptical distributions and some of its properties such as marginal and conditional distributions are reviewed in Section 2. The consistency property of marginal distributions of an elliptical family and its implications in multivariate modeling with elliptical copula is discussed in Section 3. Sampling algorithms to draw from the conditional distribution of an elliptical distribution are presented in Section 4. These algorithms are illustrated in a simplified example from a hydrological application in Section 5. A discussion concludes in Section 6.

2. Elliptical Distribution

A p-dimensional random vector $\mathbf{X}$ has an elliptically contoured distribution with a $p \times 1$ location vector μ and $p \times p$ dispersion matrix if it can be expressed as

$$\mathbf{X} = \mu + R_p \mathbf{A} \mathbf{U}_p, \tag{2}$$

where $R_p \geq 0$ is a random variable known as the generating variate, $\mathbf{A}$ is a $p \times p$ matrix such that $\mathbf{A}\mathbf{A}^\top = \Sigma$, and $\mathbf{U}_p$ is a p-variate random vector independent of R_p, uniformly distributed on the p-dimensional unit sphere $\mathbb{S}_p = \{(u_1, \ldots, u_p) \in \mathbb{R}^p : \sum_{i=1}^p u_i^2 = 1\}$.

If the density of R_p exists and Σ is positive definite, the density function h of $\mathbf{X}$ can be expressed with a scalar function g_p called density generator:

$$h(\mathbf{x}; \mu, \Sigma, g_p) = |\Sigma|^{-1/2} g_p\{(\mathbf{x} - \mu)^\top \Sigma^{-1} (\mathbf{x} - \mu)\}, \tag{3}$$

where g_p is uniquely determined by the distribution of R_P. Let f_{R_p} be the density of R_p. The relationship between g_p and f_{R_p} is (Fang et al., 1990, Theorem 2.9)

$$f_{R_p}(v) = \frac{2\pi^{p/2}}{\Gamma(p/2)} v^{p-1} g_p(v^2), \qquad v > 0. \tag{4}$$

In other words, the density generator function g_p in (4) must satisfy the condition

$$\int_0^\infty \frac{\pi^{p/2}}{\Gamma(p/2)} v^{p/2-1} g_p(v) \mathrm{d}v = 1, \quad v \geq 0,$$

where the integrand is the density of R_p^2. The distribution of $\mathbf{X}$ is denoted by $\mathscr{E}_p(\mu, \Sigma, g)$. More details about elliptical distributions can be found in Fang et al. (1990).

Table 1 summarizes the density generator g_p and generating variate R_p for some examples of elliptical families. The most commonly used examples are the multivariate normal distribution and the multivariate Student t distribution. For the multivariate normal distribution, the density generator $g_p(u)$ is $c_p \exp(-u/2)$ with normalizing constant c_p, and the generating variate R_p is a chi variable with p degrees of freedom. For the multivariate Student t distribution with ν degrees of freedom, the density generator $g_p(u) = c_{p,\nu}(1 + u/\nu)^{-(p+\nu)/2}$ with normalizing constant $c_{p,\nu}$, and the generating variate R_p is such that R_p^2/p follows an F distribution with parameters p and ν, denoted by $F(p, \nu)$. Another popular example is the exponential power family, also known as the generalized normal or generalized error distribution, which has been widely used in practice when multivariate normality is rejected (e.g., Lindsey, 1999; Basu et al., 2001). Its density generator is, with $\gamma > 0$ and $s > 0$, $g_p(u) = C_{p,\gamma,s} \exp(-\gamma u^s)$, which facilitates the derivation of the distribution of its generating variate — $R_p^{2s}\gamma$ follows a $\Gamma\big(p/(2s), 1\big)$ distribution. It reduces to the multivariate normal distribution when $s = 1$ and to a form of multivariate Laplace distribution when $s = 1/2$. Heavier or lighter tails than the normal distribution is achieved by taking $s < 1$ or $s > 1$. Note that the Kotz type family and the Pearson Type VII family are very wide: the former covers the multivariate normal family and the multivariate exponential power family; the later covers the multivariate Student t family.

Elliptical distributions are often characterized by characteristic functions in many references (e.g., Fang et al., 1990). Without loss of generality, consider the case with $\mu = \mathbf{0}$ and $\Sigma = \mathbf{I}_p$, the identity matrix of dimension p. The characteristic function of $\mathscr{E}_p(\mathbf{0}, \mathbf{I}_p, g)$ is

$$\int_{\mathbb{R}^p} \exp(i\mathbf{t}^\top \mathbf{x}) g_p(\mathbf{x}^\top \mathbf{x}) \mathrm{d}\mathbf{x} = \phi_p(\mathbf{t}^\top \mathbf{t})$$

for some scalar function ϕ_p known as characteristic generator (e.g., Cambanis et al., 1981). It can be used to check the marginal consistency of elliptical distribution, especially for the stable laws

TABLE 1. *Density generator g_p and generating R_p for some p-dimensional elliptical families and conditional sampling of the generating variate $R_{r,p,q(x_2)}$ for a r-dimensional subvector $\mathbf{X}_1$ given the rest $\mathbf{X}_2 = \mathbf{x}_2$. The normalizing constant C in each generator depends on its subscripts. BeII(a,b) means a Beta distribution of type II with parameters a and b.*

Type	Density generator $g_p(u)$	Sampling of R_p	Conditional Sampling of $R_{r,p,q(x_2)}$
Kotz type ($\gamma > 0, s > 0, 2\nu + p > 2$)	$C_{p,\gamma,s,\nu} u^{\nu-1} \exp(-\gamma u^s)$	$\nu + p/2 > 1 + s$: $R_p^{2s} r \sim \Gamma\big((\nu + p/2 - 1 - s)/s, 1\big)$; otherwise nonstandard	Nonstandard
Multivariate normal	$C_p \exp(-u/2)$	$R_p^2 \sim \chi^2(p)$	$R^2_{r,p,q(\mathbf{x}_2)} \sim \chi^2(r)$
Exponential power ($\gamma > 0$, $s > 0$)	$C_{p,\gamma,s} \exp(-\gamma u^s)$	$p/2 > s$: $R_p^{2s}\gamma \sim \Gamma\big((p/2 - s)/s, 1\big)$; otherwise nonstandard	Nonstandard
Pearson type VII ($\nu > p/2$, $s > 0$)	$c_{p,\nu,s}(1 + u/s)^{-\nu}$	$R_p^2/s \sim \mathrm{BeII}(p/2, \nu - p/2)$	$R^2_{r,p,q(\mathbf{x}_2)}/\{s + q(\mathbf{x}_2)\} \sim \mathrm{BeII}(r/2, \nu - r/2)$
Multivariate Student-t ($\nu \in \mathbb{N}$)	$C_{p,\nu}(1 + u/\nu)^{-(p+\nu)/2}$	$R_p^2/p \sim F(p, \nu)$	$R^2_{r,p,q(\mathbf{x}_2)}/\{\nu + q(\mathbf{x}_2)\} \sim \mathrm{BeII}\big(r/2, (p + \nu - r)/2\big)$
Pearson type II ($\nu > -1$)	$C_{p,\nu}(1 - u)^{\nu}$, $u \in (0, 1)$	$R_p^2 \sim \mathrm{Beta}(p/2, \nu + 1)$	$R^2_{r,p,q(\mathbf{x}_2)}/\{1 - q(\mathbf{x}_2)\} \sim \mathrm{Beta}(r/2, \nu + 1)$
Logistic	$C_p \exp(-u)/\big(1 + \exp(-u)\big)^2$	$p = 2$: $R_2^2 \sim \mathrm{Logistic}(0, 1)$ truncated to $\mathbb{R}^+$; otherwise nonstandard	$p = 2$, $r = 1$: $R^2_{1,q(\mathbf{x}_2)} \sim \mathrm{Logistic}\big(-q(\mathbf{x}_2), 1\big)$ truncated to $\mathbb{R}^+$; otherwise nonstandard

family which is often defined through its characteristic functions because the densities are usually not available in explicit forms (e.g., Nolan, 2006).

Sampling from an elliptical distribution is straightforward from the stochastic representation (2) if sampling of the generating variate R_p is known. Generation of uniform variate $\mathbf{U}_p$ on the unit sphere $\mathbb{S}_p$ can be done by $\mathbf{U}_p = \mathbf{Z}_p/\|\mathbf{Z}_p\|$, where $\mathbf{Z}_p$ is $N(\mathbf{0}, \mathbf{I}_p)$ (Marsaglia, 1972). Generation of R_p can often be done by sampling a transformation of it such as R_p^2 and then transforming back; see column 3 in Table 1. When neither R_p nor any transformation of it has a standard distribution with a readily usable random number generator, sampling from f_{R_p} has to be done with general random variate generation techniques (e.g., Devroye, 1986).

To construct a meta-elliptical distribution with given marginal distributions and an elliptical copula, it suffices to consider only $\mathbf{X} \sim \mathscr{E}_p(\mathbf{0}, \Sigma, g)$, with Σ being a correlation matrix, because copulas are invariant with respect to monotone transformations. The implicit elliptical copula of $\mathbf{X}$ is

$$C(u_1, \ldots, u_p) = G\{G_1^{-1}(u_1), \ldots, G_p^{-1}(u_p)\}, \qquad u_i \in (0,1), \quad i = 1, \ldots, p, \tag{5}$$

where G is the multivariate CDF of $\mathbf{X}$, G_i is the univariate marginal CDF of $\mathbf{X}_i$, and G_i^{-1} is the inverse function of G_i, $i = 1, \ldots, p$ Clearly, G_i's and G_i^{-1}'s are important in application of meta-elliptical distributions. A meta-elliptically distributed target random vector $\mathbf{Y}$ with marginal distributions $F_1, \ldots, F_p$ is obtained from

$$Y_i = F_i^{-1}\{G_i(X_i)\}, \quad i = 1, \ldots, n, \tag{6}$$

where F_i is the inverse function of F_i. An elliptical copula is a meta-elliptical distribution with all F_i's being the CDF of uniform over $(0,1)$.

3. Marginal Consistency and Its Implications

3.1. Marginal Consistency

Without loss of generality, consider an elliptically contoured random vector $\mathbf{X} \sim \mathscr{E}_p(\mathbf{0}, \mathbf{I}_p, g_p)$. Its marginal distributions are still elliptical distributions, but not necessarily in the same elliptical family. Partition $\mathbf{X}$ into $(\mathbf{X}_1, \mathbf{X}_2)$, where $\mathbf{X}_1$ is $r \times 1$, $r < p$, and $\mathbf{X}_2$ is $(p-r) \times 1$. Correspondingly, partition Σ into

$$\Sigma = \begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{pmatrix}.$$

The marginal distribution of $\mathbf{X}_1$ is $\mathscr{E}_r(\mathbf{0}, \Sigma_{11}, g_{r,p})$, where the density generator $g_{r,p}$ may depend on p (Fang et al., 1990, Theorem 2.16).

Kano (1994) defined marginal consistency as follows. An elliptical distribution family with density generator g_p has marginal consistency if and only if

$$\int_R g_{p+1}\left(\sum_{i=1}^{p+1} x_i^2\right) \mathrm{d}x_{p+1} = g_p\left(\sum_{i=1}^{p} x_i^2\right), \tag{7}$$

for all $p \in N$ and almost all $(x_1, \ldots, x_p) \in \mathbb{R}^p$. An elliptical family is marginally consistent if it possesses the marginal consistency property; it is marginally inconsistent otherwise. Equation (7) essentially requires that the marginal density of the first p components of an elliptical distribution of dimension $p+1$ is the same as the density of an elliptical distribution of dimension p in the same elliptical family. The family has marginal consistency if this holds for all $p \in N$.

The marginal consistency property is equivalent to any one of the following (Kano, 1994, Theorem 1):

1. the density generator satisfies $g_p(u) = \int_u^\infty (y-u)^{-1/2} g_{p+1}(y)$ for all $p \in N$ and almost all $u > 0$;
2. the characteristic function ϕ_p is unrelated to p;
3. the generating variate R_p can be expressed as $R_p = \chi_p/\sqrt{\xi}$, for any $p \in N$, where χ_p^2 is a chi-squared variable with p degrees of freedom, and $\xi > 0$ is a random variable unrelated to p, and ξ, χ_p and U_p are mutually independent;
4. $\mathbf{X}$ can be expressed as a mixture of standard normal random vectors with an independent mixing variable whose distribution is unrelated to p.

The first three conditions are statements of the marginal consistency property in terms of density, characteristic function, and stochastic representation. The last condition provides an intuitive way to check for marginal consistency.

The density generator and sampling distribution of R_p in Table 1 can tell which families are consistent. As pointed out by Kano (1994), the families of multivariate normal distributions, multivariate Student t distributions, and stable laws distributions have marginal consistency; the families of logistic, Pearson Type II, Pearson Type VII, Kotz type, and multivariate Bessel distributions do not have marginal consistency. The exponential power family with $s > 1$ has lighter tails than the normal distribution, which cannot be a normal mixture; the family with $s \in (0,1)$ is a scale mixture of normal distribution with a nonstandard mixing variable (Gómez-Sánchez-Manzano et al., 2008), whose distribution depends on p. Therefore, the exponential power family is inconsistent for all $s \neq 1$.

Application of elliptical families without the marginal consistency property may lead to undesired features in multivariate modeling. For instance, the univariate marginal distributions of a member from such a family would depend on p. In risk analysis, this would imply the rather odd fact that the distribution of any marginal loss depends on the number of losses considered in the portfolio (Bilodeau, 2003). For marginal distributions of dimension two or higher, this would imply that the joint distribution of a subset of losses depends on the number of losses under consideration.

3.2. Implications on Elliptical Copula Modeling

Consistent elliptical families provide elliptical copulas that have desired properties in meta-elliptical modeling. The univariate CDF G_i of such families does not depend on p, and hence does not affect the density of the elliptical copula as seen from (5). The elliptical copula of any lower order marginal multivariate distribution does not depend on p either. The two properties make software implementation easy for these copulas as long as the multivariate CDF G is available. For instance, the R package `copula` provides density, distribution, and random number generation for normal copula and t copula (Kojadinovic and Yan, 2010b) based on the implementation of multivariate normal and t distributions in the R package `mvtnorm` (Genz and Bretz, 2009).

Application of elliptical copulas coming from inconsistent elliptical families is limited in several aspects that can subtly compromise the modeling approach.

Impact of Univariate Marginal Inconsistency The effect of inconsistency on the univariate distributions is not removed by using elliptical copulas with desired univariate margins. From (5), the density of the elliptical copula depends on the marginal distribution G_i's of the elliptical family. Further, the univariate marginal densities are usually difficult to obtain explicitly for exponential power or logistic families; they are trivial only for consistent elliptical families. This means that the univariate marginal CDFs (and their densities) may not be available and differ according to p. It prohibits a universal implementation of random number generation with the "quantile-to-quantile" transformation (6) and evaluation of the density and the CDF of the meta-elliptical distribution. The likelihood or pseudo-likelihood method for parameter estimation would not be possible (e.g., Kojadinovic and Yan, 2010a). Even if the marginal density and CDF can be derived, they need to be derived for all practically possible p's, which complicates software implementation. This explains why, despite the popularity of exponential power distributions, its copula is not widely used in meta-elliptical modeling.

Impact of Marginal Copula Inconsistency An inconsistent elliptical family also has implications on the dependence structure captured by the implicit elliptical copula. For any $1 < r < p$, the elliptical copula of an r-dimensional marginal distribution of a p-dimensional elliptical distribution depends on p, and is not the same as the copula of an r-dimensional elliptical distribution from the same family, which would be the case for a marginally consistent elliptical family. In financial applications, the dependence structure of a fixed number of risks in a portfolio would change as the portfolio size increases. In our hydrological example, this would imply that the dependence structure between duration and volume of extremal storms would change if peak intensity were additionally considered. The densities of r-dimensional marginal distributions of an inconsistent p-dimensional elliptical family are hard to obtain. These distributions have known location and dispersion parameters but the density generators are complex and dependent on p. As p increases, a sequence of marginal copulas of dimension r are obtained, which all have the same location and dispersion parameters. To appreciate their differences, consider Kendall's tau and Spearmans's rho. As Kendall's tau of elliptical distributions is independent of the density generator, $\tau = 2\arcsin(r_{ij})$, where r_{ij} is the (i, j)th entry of the correlation matrix Σ, they all have the same pairwise Kendall's tau (Fang et al., 2002). Nevertheless, they have different Spearman's rho, which generally depends on both the density generator and r_{ij} (Abdous et al., 2005).

For illustration, consider the bivariate marginal copula of an exponential power family $\mathscr{E}_p(\mathbf{0}, \mathbf{I}_p, g_p)$ with dimension $p \in (2,4,8)$ and $\gamma = s = 0.5$. Sampling from the distribution is easy (see Table 1). Although the univariate marginal distributions and the bivariate marginal copulas have no explicit expressions, we can approximate them using a large sample of draws from the exponential power distribution. To facilitate comparison, we convert the margins to standard normal. Figure 1 shows the contours of bivariate kernel densities with standard normal margins and bivariate marginal copulas from exponential power distributions with $p \in (2,4,8)$. The bivariate kernel densities were obtained with the `kde2d` function in the R package `MASS` (Venables and Ripley, 2002) for a random sample of size 400,000 for each distribution. It is clear that these bivariate copulas are quite different as p changes.

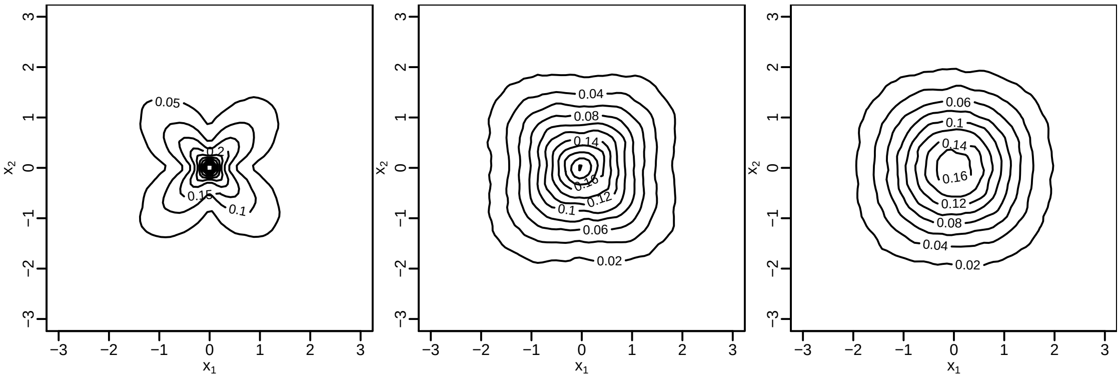

FIGURE 1. *Contours of bivariate kernel densities with standard normal margins and bivariate marginal copula of an exponential power distribution with $\gamma = s = 0.5$ and three values of p: Left: $p = 2$. Center: $p = 4$. Right: $p = 8$. The contours were obtained with a sample of 400,000 draws from each distribution.*

4. Conditional Sampling from Meta-elliptical Distribution

Conditionally sampling of a random vector $\mathbf{Y}$ with a meta-elliptical distribution can be done easily with conditional sampling from an elliptical distribution given the marginal transformation (6). Partition $\mathbf{Y}$ into $(\mathbf{Y}_1, \mathbf{Y}_2)$, where $\mathbf{Y}_1$ is $r \times 1$, $r < p$, and $\mathbf{Y}_2$ is $(p-r) \times 1$. Let $f_{\mathbf{Y}_1|\mathbf{Y}_2}(\mathbf{y}_1|\mathbf{y}_2)$ be the conditional density of $\mathbf{Y}_1$ given $\mathbf{Y}_2 = \mathbf{y}_2$. The idea is to transform $\mathbf{Y}$ to $\mathbf{X}$ in the "elliptical" space by the inverse transformation of (6), do the conditional sampling of $\mathbf{X}$, and transform the sample back to the original scale by (6). Partition accordingly $\mathbf{X}$ into $(\mathbf{X}_1, \mathbf{X}_2)$, $(F_1, \ldots, F_p)$ into $(\mathbf{F}_1, \mathbf{F}_2)$, and $(G_1, \ldots, G_p)$ into $(\mathbf{G}_1, \mathbf{G}_2)$. Let $\mathbf{x}_2 = \mathbf{G}_2^{-1}\{\mathbf{F}_2(\mathbf{y}_2)\}$. Let $f_{\mathbf{X}_1|\mathbf{X}_2}(\mathbf{x}_1|\mathbf{x}_2)$ be the conditional density of $\mathbf{X}_1$ given $\mathbf{X}_2 = \mathbf{x}_2$. If $\mathbf{X}_1$ is a draw from $f_{\mathbf{X}_1|\mathbf{X}_2}(\mathbf{x}_1|\mathbf{x}_2)$, then $\mathbf{F}_1^{-1}\{\mathbf{G}_1(\mathbf{X}_1)\}$ is a draw from $f_{\mathbf{Y}_1|\mathbf{Y}_2}(\mathbf{y}_1|\mathbf{y}_2)$. Therefore, the problem boils down to sampling from $f_{\mathbf{X}_1|\mathbf{X}_2}(\mathbf{x}_1|\mathbf{x}_2)$.

The conditional distribution of an r-dimensional vector $\mathbf{X}_1$ given $\mathbf{X}_2 = \mathbf{x}_2$ is still an elliptical distribution but not in the same family as $\mathbf{X}$ (Fang et al., 1990, Theorem 2.18). In copula modeling we have $\mu = \mathbf{0}$, and the conditional distribution is $\mathscr{E}_r(\mu_{1.2}, \Sigma_{11.2}, g_{r,p,q(\mathbf{x}_2)})$, where $\mu_{1.2} = \Sigma_{12}\Sigma_{22}^{-1}\mathbf{x}_2$, $\Sigma_{11.2} = \Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21}$, $q(\mathbf{x}_2) = \mathbf{x}_2\Sigma_{22}^{-1}\mathbf{x}_2$, and the density generator $g_{r,p,a}$ is

$$g_{r,p,a}(u) = \frac{\Gamma(r/2) g_p(a+u)}{\pi^{r/2} \int_0^\infty v^{r/2-1} g_p(a+v) \mathrm{d}v}. \tag{8}$$

Given that the marginal distribution of $\mathbf{X}_2$ may depend on p, it is not surprising that the density generator of the conditional distribution depends on p in addition to $q(\mathbf{x}_2)$ and r.

We present two conditional sampling approaches. The first one is based on the stochastic representation of elliptical distributions because the conditional distribution of $\mathbf{X}_1$ given $\mathbf{X}_2 = \mathbf{x}_2$ is still an elliptical distribution. The second one is an acceptance-rejection approach with the proposal distribution being the marginal distribution recentered at the conditional location parameter. The second approach also provides a weighted sample, which can be used in importance sampling to estimate quantities that are of an integral form such as mean or variance.

Stochastic Representation Method Since the conditional distribution $f_{\mathbf{X}_1|\mathbf{X}_2}$ is an elliptical distribution, sampling can be done in principle with the stochastic representation method. The difficulty is how to sample the generating variate $R_{r,p,q(\mathbf{x}_2)}$ whose distribution, similar to $g_{r,p,q(\mathbf{x}_2)}$, may depend on p in addition to $q(\mathbf{x}_2)$. Putting (8) into (4) with dimension r gives the density of $R_{r,p,q(\mathbf{x}_2)}$

$$f_{R_{r,p,q(\mathbf{x}_2)}}(v) = \frac{2\pi^{r/2}}{\Gamma(r/2)} v^{r-1} g_{r,p,q(\mathbf{x}_2)}(v^2), \qquad v > 0, \tag{9}$$

which implicitly depends on p for marginally inconsistent elliptical families. Conditional sampling from $f_{\mathbf{X}_1|\mathbf{X}_2}$ is straightforward when $R_{r,p,q(\mathbf{x}_2)}$ can be sampled; see column 4 in Table 1. Of course, for some families such as normal and Student t, the conditional distribution is known and standard methods can be employed instead of the stochastic representation method.

Conditional sampling is more challenging when $R_{r,p,q(\mathbf{x}_2)}$ or a transformation of it does not have a standard distribution, as is the case for marginally inconsistent families in general. The density generator $g_{r,p,q(\mathbf{x}_2)}$ involves an integral in the normalizing constant, which may not have a closed-form evaluation. The density $f_{R_{r,p,q(\mathbf{x}_2)}}(v)$ in (9) is only known up to a normalizing constant in general. Therefore, one can apply the adaptive rejection Metropolis sampling (ARMS) algorithm (Gilks et al., 1995), which samples from an arbitrary density function known up to a constant with convex support. This algorithm is available in the R package HI (Petris et al., 2006). The draws from the algorithm may be autocorrelated, but can still be used for inferences as in Markov chain Monte Carlo (MCMC) methods. Autocorrelation in the sample may be alleviated by obtaining a larger initial sample and then thinning it.

Acceptance-Rejection Method Using the rejection method to sample from $f_{\mathbf{X}_1|\mathbf{X}_2}$ requires evaluation of $f_{\mathbf{X}_1|\mathbf{X}_2}(\mathbf{x}_1|\mathbf{x}_2)$ and a proposal density $\pi(\mathbf{x}_1)$ such that $f_{\mathbf{X}_1|\mathbf{X}_2}(\mathbf{x}_1|\mathbf{x}_2) \le M\pi(\mathbf{x}_1)$ for some $0 < M < \infty$. The bigger M, the less efficient rejection algorithm. The conditional density $f_{\mathbf{X}_1|\mathbf{X}_2}(\mathbf{x}_1|\mathbf{x}_2)$ is $h_r(\mathbf{x}_1; \mu_{1.2}, \Sigma_{11.2}, g_{r,p,q(\mathbf{x}_2)})$, which by definition, is calculated as the ratio between the full density $h_p(\mathbf{x}; \mu, \Sigma, g_p)$ and the marginal density $h_{p-r}(\mathbf{x}_2; \mu_2, \Sigma_{22}, g_{p-r,p})$ for some generator function $g_{p-r,p}$ depending on p. From the discussion about marginal consistency, this method can only be easily carried out for marginally consistent families, where the marginal densities can be easily obtained. The marginal distribution of $\mathbf{X}_1$ re-centered at $\mu_{1.2}$ provides a natural proposal distribution with density $h_r(\mathbf{x}_1; \mu_{1.2}, \Sigma_{11}, g_{r,p})$ for some generator $g_{r,p}$. This proposal density has a heavier tail than the target density. The smallest M that satisfies the constraints on the proposal density is the ratio of two density evaluated at the center $\mu_{1.2}$:

$$M = h_r(\mu_{1.2}; \mu_{1.2}, \Sigma_{11.2}, g_{r,p,q(\mathbf{x}_2)}) / h_r(\mu_{1.2}; \mu_{1.2}, \Sigma_{11}, g_{r,p}).$$

The calculation of M is straightforward using (3) for marginally consistent families. When sampling from the proposal distribution $\mathscr{E}_r(\mu_{1.2}, \Sigma_{11}, g_{r,p})$ is possible (e.g., from the stochastic representation approach), Algorithm 1 can be used to sample from $f_{\mathbf{X}_1|\mathbf{X}_2}(\mathbf{x}_1|\mathbf{x}_2)$. In the example of the multivariate normal or Student t distribution, sampling from $\mathscr{E}_r(\mu_{1.2}, \Sigma_{11}, g_{r,p})$ is easy with the R package `mvtnorm` (Genz et al., 2011).

Application of the acceptance-rejection method is limited by its need of the marginal density and random number generation of $\mathbf{X}_1$. Random number generation of $\mathbf{X}_1$ is easy as long as a random number generator of the whole vector $\mathbf{X}$ is available. The marginal density of $\mathbf{X}_1$ may

Algorithm 1 Acceptance-Rejection Sampling from $g(\mathbf{x}_1|\mathbf{x}_2)$.

1: **repeat**
2: Generate $\mathbf{z}$ from $\mathscr{E}_r(\mu_{1.2}, \Sigma_{11}, g_{r,p})$
3: Generate u from $U(0,1)$.
4: **until** $u < h_r(\mathbf{z}; \mu_{1.2}, \Sigma_{11.2}, g_{r,p,q(\mathbf{x}_2)}) / \{M h_r(\mathbf{z}; \mu_{1.2}, \Sigma_{11}, g_{r,p})\}$
5: **return** $\mathbf{z}$

be difficult to derive if the distribution of $\mathbf{X}$ is not a consistent elliptical family. The stochastic representation method is more applicable in that it only needs random number generation from a univariate random variable $R_{r,p,q(\mathbf{x}_2)}$, whose density is known up to a constant if the density generator g_p of $\mathbf{X}$ is known.

Importance Sampling When the goal of conditional sampling is to approximate the conditional expectation of a function H of $\mathbf{X}_1$ given $\mathbf{X}_2 = \mathbf{x}_2$, $E[H(\mathbf{X}_1)|\mathbf{X}_2 = \mathbf{x}_2]$, importance sampling can be adapted from the acceptance-rejection algorithm (e.g., Robert and Casella, 2004). Unlike acceptance-rejection sampling, importance sampling uses all the generated values $\mathbf{z}$ without rejection and assigns an importance weight to each one of them. Let $\mathbf{z}_1, \ldots, \mathbf{z}_n$ be a large sample of size n from the proposal distribution $\mathscr{E}_r(\mu_{1.2}, \Sigma_{11}, g_{r,p})$. The target conditional expectation is estimated by

$$\hat{H}_{\mathbf{x}_2} = \frac{\sum_{i=1}^n H(\mathbf{z}_i) w(\mathbf{z}_i)}{\sum_{i=1}^n w(\mathbf{z}_i)},$$

where the importance weight is given by $w(\mathbf{z}_i) = h_r(\mathbf{z}_i; \mu_{1.2}, \Sigma_{11.2}, g_{r,p,q(\mathbf{x}_2)}) / h_r(\mathbf{z}_i; \mu_{1.2}, \Sigma_{11}, g_{r,p})$. Because the weights are known, the standard error of $\hat{H}_{\mathbf{x}_2}$ is easily calculated as

$$\mathrm{SE}(\hat{H}_{\mathbf{x}_2}) = \frac{\sqrt{\sum_{i=1}^n \{H(\mathbf{z}_i) - \tilde{H}_{\mathbf{x}_2}\}^2 w^2(\mathbf{z}_i)}}{\sum_{i=1}^n w(\mathbf{z}_i)},$$

where $\tilde{H}_{\mathbf{x}_2} = \sum_{i=1}^n H(\mathbf{z}_i)/n$. When the stochastic representation method is available, the target density itself can be used as the proposal density. In that case, all importance weights are equal to one, which lead to the usual Monte Carlo estimator $\tilde{H}_{\mathbf{x}_2}$ with the usual standard error $\mathrm{SE}(\tilde{H}_{\mathbf{x}_2}) = [\sum_{i=1}^n \{H(\mathbf{z}_i) - \tilde{H}_{\mathbf{x}_2}\}^2]^{-1/2}/n$.

5. An Illustration

We illustrate conditional sampling with a hydrological example where a meta-elliptical model with a t copula. Since the multivariate t distribution is a marginally consistent, both the stochastic representation method and the importance sampling method can be easily applied. We compare their efficiency in approximating a univariate conditional mean and a bivariate conditional probability.

In hydrology, estimation of design storms and the associated risks requires detailed knowledge of three major storm characteristics: volume, duration, and peak intensity. A thorough theoretical and practical introduction concerning the use of copulas in hydrology can be found in Salvadori et al. (2007). We use the fitted model from Wang et al. (2010) to illustrate risk assessment with conditional sampling. The model was a multivariate distribution with a Student t copula for volume (Y_1, 0.01 inch), duration (Y_2, hour), and peak intensity (Y_3, 0.01 inch / 15 min) of annual

TABLE 2. *Summaries of estimated conditional expectation of peak intensity given duration and volume with two methods from 1000 replicates, each based on a sample of 10,000. Estimator $\tilde{H}$ is based on a sample directly from the conditional distribution. Estimator $\hat{H}$ is based on a weighted sample from a proposal distribution. SE: Mean of the standard error. SEE: Empirical standard deviation.*

Duration	Volume	Estimator $\tilde{H}$			Estimator $\hat{H}$		
		Mean	SE	SEE	Mean	SE	SEE
8	100	9.913	0.068	0.068	9.914	0.089	0.089
8	200	11.231	0.046	0.046	11.228	0.039	0.038
8	300	12.486	0.050	0.050	12.485	0.043	0.042
8	400	13.687	0.060	0.058	13.687	0.054	0.054
8	500	14.929	0.073	0.072	14.933	0.074	0.075
5	200	13.147	0.053	0.052	13.146	0.046	0.046
10	200	10.411	0.044	0.045	10.411	0.037	0.038
15	200	9.033	0.041	0.041	9.034	0.035	0.034
20	200	8.122	0.041	0.040	8.122	0.036	0.036
25	200	7.441	0.042	0.042	7.440	0.040	0.040

extreme rainfall based on the 15-min precipitation data at a monitoring station in Connecticut. The extreme rainfall storm in a given year was defined to be the rainfall event which possesses the largest empirical joint cumulative probability of volume and peak intensity (Kao and Govindaraju, 2007). The fitted marginal distribution was $\mathrm{LN}(5.573, 0.430)$ for Y_1, $\mathrm{Gamma}(2.000, 6.747)$ for Y_2, and $\mathrm{LN}(2.292, 0.515)$ for Y_3, where $\mathrm{LN}(a,b)$ is a lognormal distribution with mean a and standard deviation b on the log scale, and $\mathrm{Gamma}(a,b)$ is a gamma distribution with shape a and scale b. The copula was a t-copula with 5 degrees of freedom (df) and dispersion matrix

$$\Sigma = \begin{pmatrix} 1.000 & 0.541 & -0.083 \\ 0.541 & 1.000 & -0.463 \\ -0.083 & -0.463 & 1.000 \end{pmatrix}.$$

Consider the conditional expectation of peak intensity at a set of combinations of volume and duration: $E(Y_3|Y_1 = y_1, Y_2 = 8)$ for $y_1 \in \{100, 200, 300, 400, 500\}$ and $E(Y_3|Y_1 = 200, Y_2 = y_2)$ for $y_2 \in \{5, 10, 15, 20, 25\}$. We approximate the conditional expectations by Monte Carlo simulation. Let F_i be the CDF of Y_i, $i = 1, 2$ and 3, and let G_1 be the CDF of univariate t distribution with df 5. Because the multivariate Student t family possesses marginal consistency, the three marginal distributions are all G_1. For $i = 1$ and 2, let $x_i = G_1^{-1}\{F_i(y_i)\}$. Define function $H(z) = F_3^{-1}\{G_1(z)\}$. The target conditional expectation is then $E[H(X_3)|X_1 = x_1, X_2 = x_2]$, where $(X_1, X_2, X_3)^\top$ follows a trivariate Student t distribution centered at $\mathbf{0}$, with dispersion matrix Σ and df 5. We compare two approximations. The first one, denoted by $\tilde{H}$, is based on a sample from the conditional distribution of X_3 given $(X_1, X_2) = (x_1, x_2)$ using the stochastic representation method. The second one, denoted by $\hat{H}$, is based on a weighted sample drawn from the proposal distribution defined in the acceptance-rejection method, which is the marginal distribution of X_3 recentered at the conditional mean. Each approximation is based on a sample of size $n = 10,000$ and we repeat each approximation 1000 times.

Table 2 summarizes the mean of the estimates, mean of the standard errors, and empirical standard error of the estimates from both approximation methods. The means of the estimates of the two methods are very close for all target conditional expectations. The conditionally expected peak intensity increases with volume when duration is fixed at 8 (hour), and decreases with duration when volume is fixed at 200 (0.01 inch). From the empirical standard errors, estimator $\hat{H}$ is as efficient as or slightly more efficient than $\tilde{H}$ except for $(x_1, x_2) = (100, 8)$. Finally, the close agreement between the means of the standard errors and the empirical standard errors for $\hat{H}$ suggests that the standard error formula for the weighted sample mean provides a good variation measure of the estimate.

For further illustration on joint events, consider $\Pr(Y_2 > y_2, Y_3 > y_3 | Y_1 = y_1)$ for $y_2 = 26.244$, $y_3 = 19.144$, and $y_1 = 500$. This is the conditional probability that the duration exceeds 26.244 (hours) and the peak intensity exceeds 19.144 (0.01 inch / 15 min) simultaneously given that the volume is 500 (0.01 inch). These thresholds are the 90th percentiles of the marginal distributions of duration and peak intensity, respectively. The marginal conditional probabilities $\Pr(Y_2 > y_2 | Y_1 = y_1)$ and $\Pr(Y_3 > y_3 | Y_1 = y_1)$ are approximated as 0.310 and 0.106, respectively. If duration and peak intensity were conditionally independent, $\Pr(Y_2 > y_2, Y_3 > y_3 | Y_1 = y_1)$ would be approximately 0.033. We approximate this conditional probability with a large sample from the bivariate conditional distribution of duration and peak intensity given volume at 500 (0.01 inch). Figure 2(a) shows a sample of 1000 observations drawn from the conditional distribution. Negative dependence between duration and peak intensity given volume at 500 is clearly observed. Figure 2(b) shows a sample of 1000 observations drawn from the importance sampler. Apparently, the importance sampler has heavier tails at both margins than the target distribution. We approximated the target probability with a sample of size $n = 100,000$ for both methods and repeated the process for 1000 times. The mean of the approximation from both methods are very close, 0.010, much lower than what would happen under conditional independence (0.033). This is not surprising given the negative dependence between duration and peak intensity given volume at 500. The standard errors of the two methods are very close, 3.27×10^{-4} from direct samples and 3.21×10^{-4} from weighted samples.

6. Discussion

Meta-elliptical distributions based on elliptical copulas have been a useful tool for multivariate modeling in many fields. In addition to basic properties, their tail properties have been studied as motivated by the importance of tail dependence in practice (e.g., Frahm et al., 2003; Landsman and Valdez, 2003; Hashorva, 2008; Manner and Segers, 2011). Nevertheless, the lesser known marginal consistency property of elliptical distributions and its implications in meta-elliptical modeling appear to have gained insufficient attention. The impact of inconsistent elliptical families on elliptical copula modeling had puzzled us in our own experience. We hope that this note can help practitioners of meta-elliptical distributions to avoid misuse of elliptical copulas.

Conditional sampling from multivariate distributions is a necessity for problems where no closed-form solution is available. For Archimedean copulas, sampling algorithms are often based on conditional sampling; see Hofert (2008) and Hofert (2011) for recent development. Conditional sampling of meta-elliptical distributions boils down to conditional sampling from elliptical distributions, which may not be well known to practitioners using elliptical copulas. The marginal consistency property also has implications on conditional distributions of meta-elliptical distributions. In our opinion, the stochastic representation approach is generally applicable since it

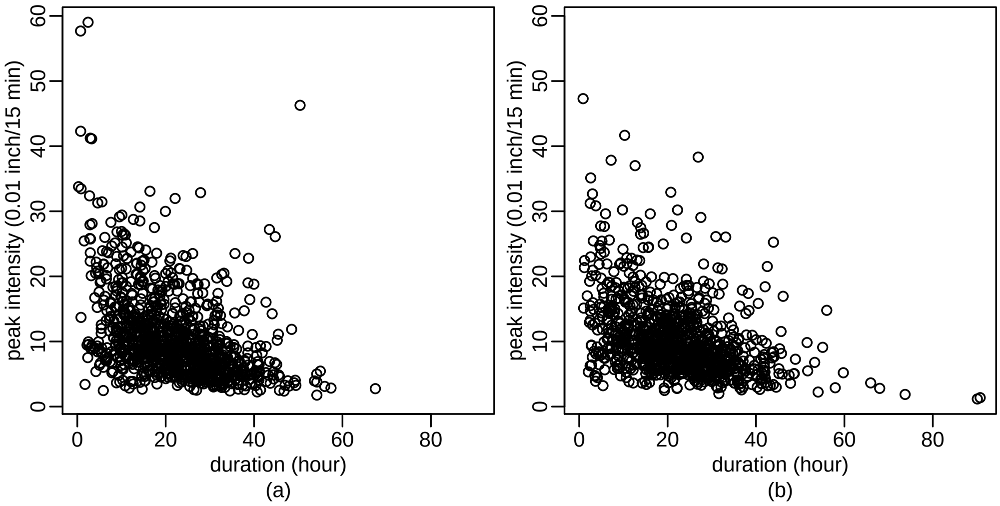

FIGURE 2. *(a) 1000 draws from the conditional distribution of duration and peak intensity given volume at 2 inch. (b) 1000 draws from the importance sampler for the conditional distribution of duration and peak intensity given volume at 2 inch.*

only needs the density generator of the whole vector and sampling of a univariate density known up to a constant. The acceptance-rejection method (and the importance sampling method) requires the density generator and sampling of the subvector in addition to the density generator of the whole vector.

Acknowledgments

The authors are grateful to three anonymous reviewers and the guest editor for their constructive comments. They thank Mr. Ellis Schaffer for carefully proofreading the manuscript.

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